ALGEBRAIC TOPOLOGY-MATH 7020

Lecture Notes by Rafal Komendarczyk and Shahriyar Roshan Zamir

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1. Introduction

In the words of the great Carly Simon " You probably think this song is about you. It aint!". (The second part isn't part of the song.) That line was for myself, this introduction or course is not about me. However, I would like to recall two short stories for your amusement.

The first time I encountered Betti numbers was during the 2017 CA+ conference when I was a first year Master's student at the University of Minnesota-Duluth. I could not fathom who Betti was and why everybody wanted Betti's number! Roughly 10 years later, I know how to compute certain betti numbers, even graded ones, but it was not until writing this introduction that I finally looked into who Enrico Betti was.

Perhaps it is a little bit of mathematical karma that when I took this course, as a first year Ph.D student, the class was taught by a postdoc. Six years later I find myself in the privileged position of teaching you this course; a privilege I shall not take for granted.

The notes follow Dr. Rafal Komendarczyk's hand written notes closely. Occasionally, I have diverged by adding extra examples or proofs that were not part of the original notes. In particular, I added additional proofs in section 1 and greatly expanded the section on the local degree formula (section 12). The two main sources for this course are Allen Hatcher's Algebraic Topology and James Munkeres' Elements of Algebraic Topology. These notes are guaranteed to have typos, please use at your own risk. Please feel free to sent corrections to me at sroshanzamir@tulane.edu. Out of stubbornness, or "principle" as I like to call it!, I did not use the help of an AI system in typing the notes.

I really enjoyed teaching the course in the order of Dr. Komendarczyk's notes. The sequence in which his notes were written made the most sense to me and the students and I highly recommend it to anyone teaching algebraic topology in the future. Moreover, a special thanks to Dr. Komendarczyk for all of his help throughout the semester and many hours of conversation. I thank my students Harsh Mishra, Shehneer Khan, Isabel Pfaff and Amna Farooq for their patience in learning the topic from a non-expert. Their many excellent questions and astute observations kept me honest and enhanced everyone's understanding of the course. In particular, many thanks to Isabel Pfaff for catching numerous typos.

Finally, I have stylistically borrowed from my good friend and mentor Dr. Joe Gallian by adding little quotes at the beginning of each day's notes. Most of these quotes are by artist, writers, and influential people I find interesting. The quotes are meant for your amusement because after all, this song is about you.

"And remember this: the page you are looking at now, I once typed the words with care with you in mind under a yellow light with the radio on."

-Charles Bukowski

2. Algebra Review

Monday January 12:
"I can't be a pessimist, because I'm alive. To be a pessimist means that you have agreed that human life is an academic matter, so I'm forced to be an optimist. I am forced to believe that we can survive whatever we must survive."

-James Baldwin

2.1. **Direct Products and Sums.** Let $G = \{G_\alpha\}_{\alpha \in I}$ be a collection of groups indexed by some set I. In Definition 1 we will define the "category-theoretic" definition of the direct product althouh we won't use that language. Moreover, in your homework you will show this is identifiable with the usual direct product e.g. [Dum99, p.152].

Definition 1. A set theoretic description of $\prod_\alpha G_\alpha$ is

$$\prod_\alpha G_\alpha = \{f : I \to \cup_{\alpha \in I} G_\alpha \mid f(\beta) \in G_\beta \text{ for all } \beta \in I\},$$

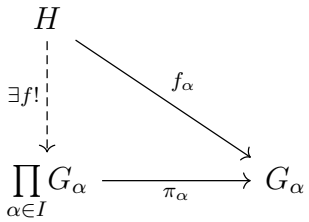
which is endowed with the following operation for all $f, h \in \prod_\alpha G_\alpha$:

$$(f * h) : I \to \cup_{\alpha \in I} G_\alpha$$
$$(f * h)(\beta) := f(\beta) \cdot_{G_\beta} h(\beta)$$

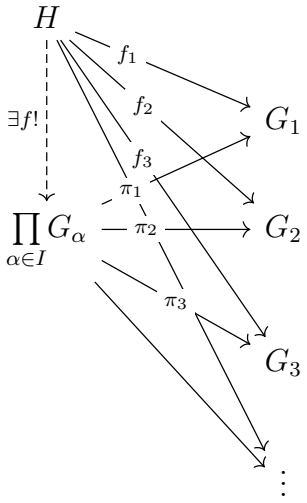
Note $f(\beta) \cdot_{G_\beta} h(\beta)$ makes sense as both $f(\beta), h(\beta) \in G_\beta$. The subscript is there to remind us which operation are we using. As currently written the product $\prod_{\alpha \in I} G_\alpha$ only describes the product as a set. One can actually show that $(\prod_\alpha G_\alpha, *)$ is in fact a group. Moreover, $\prod_{\alpha \in I} G_\alpha$ comes with natural projection maps, for each $\beta \in I$ the β-th projection map is defined as:

$$\pi_\beta : \prod_{\alpha \in I} G_\alpha \to G_\beta$$
$$\pi_\beta(f) \to f(\beta)$$

The product enjoys the following universal property: given another group, say H and a collection of maps $f_\alpha : H \to G_\alpha$ for each $\alpha \in I$, there exists a unique map f such that the following diagram commutes for any α:



If one were to expand this diagram for a countable indexing set I then it'd look something like the following where each triangle commutes.



We will encounter other universal properties throughout this course so it is a good idea to contemplate this a little bit.

Definition 2. The direct sum of G, denoted $\bigoplus_{\alpha\in I} G_\alpha$, has the following set theoretic description:

$$\begin{aligned}\bigoplus_{\alpha\in I} G_\alpha &= \{f : I \to \cup_{\alpha\in I} G_\alpha \mid f(\beta) \in G_\beta \text{ for all } \beta \in I,\ f(\beta) \neq 1_{G_\beta} \text{ for finitely many } \beta\},\\ &= \{f : I \to \cup_{\alpha\in I} G_\alpha \mid f(\beta) \in G_\beta \text{ for all } \beta \in I,\ f(\beta) = 1_{G_\beta} \text{ for all but finitely many } \beta\}.\end{aligned}$$

where 1_{G_β} denotes the identity element of the group G_β. The set $\bigoplus_{\alpha\in I} G_\alpha$ is endowed with the following operation for all $f, h \in \bigoplus_{\alpha\in I} G_\alpha$:

$$\begin{aligned}(f * h) &: I \to \cup_{\alpha\in I} G_\alpha\\ (f * h)(\beta) &:= f(\beta) \cdot_{G_\beta} h(\beta)\end{aligned}$$

It is clear that $\bigoplus_{\alpha\in I} G_\alpha \subseteq \prod_{\alpha\in I} G_\alpha$. The elements of $\prod_{\alpha\in I} G_\alpha$ are all maps with no constraints (except being in the correct target) and $\bigoplus_{\alpha\in I} G_\alpha$ are those maps such that for all but finitely many indices the output is **not** equal to the identity of the target. In general the direct sum and product need not be equal. Let us see some examples where $\bigoplus_{\alpha\in I} G_\alpha \subsetneq \prod_{\alpha\in I} G_\alpha$.

Example 1. Consider the countable collection of groups $G = \{\mathbb{Z}_2\}_{\alpha\in\mathbb{N}}$. Using the "usual definition" we see that

$$\left|\bigoplus_{\alpha\in\mathbb{N}} \mathbb{Z}_2\right| = \aleph_0,$$

where $\aleph_0$ (pronounced a-leph zero) is the first cardinal number i.e. the smallest kind of infinity which is countably infinite. This is because the direct sum of G consists of "countable tuples of entries", we only have two choices for each entry and all but finitely many entries are non-zero. One can construct a set-theoretic bijection between $\bigoplus_{\alpha\in\mathbb{N}} \mathbb{Z}_2$ and $\mathbb{N}$. However,

$\prod_{\alpha\in\mathbb{N}} \mathbb{Z}_2$ is all possible binary sequence the cardinaly of which is

$$\left|\prod_{\alpha\in\mathbb{N}} \mathbb{Z}_2\right| = 2^{\aleph_0}$$

But $2^{\aleph_0}$ is uncountably infinite therefore $\bigoplus_{\alpha\in\mathbb{N}} \mathbb{Z}_2 \subsetneq \prod_{\alpha\in\mathbb{N}} \mathbb{Z}_2$.

Example 2. Consider the collection $G = \{\mathbb{R}\}_{\alpha\in\mathbb{R}}$. This is an uncountable collection of the group $\mathbb{R}$ (which itself is uncountable.) Note that

$$\prod_{\alpha\in\mathbb{R}} \mathbb{R} = \{f : \mathbb{R} \to \mathbb{R}\}, \text{ and } \left|\prod_{\alpha\in\mathbb{R}} \mathbb{R}\right| = \mathbb{R}^{\mathbb{R}}.$$
$$\bigoplus_{\alpha\in\mathbb{R}} \mathbb{R} = \{f : \mathbb{R} \to \mathbb{R} \mid f(x_i) \neq 0 \text{ for finitely many real numbers } x_i,\ 1 \leq i \leq k\}$$

Notice the direct sum can be identified with the following set of formal sums

$$\left\{\sum_{i=1}^{k} \alpha_i x_i \mid \alpha_i, x_i \in \mathbb{R},\ \alpha_i \neq 0 \text{ for all } i\right\}$$

where the explicit map between this set and $\bigoplus_{\alpha\in\mathbb{R}} \mathbb{R}$ is given by

$$\sum_{i=1}^{k} \alpha_i x_i \longleftrightarrow f : \mathbb{R} \to \mathbb{R} \text{ where } f(x) = \begin{cases} \alpha_i & \text{if } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

In your homework you will prove a more general fact which implies

$$\bigoplus_{\alpha\in\mathbb{R}} \mathbb{R} \subsetneq \prod_{\alpha\in\mathbb{R}} \mathbb{R}$$

Remark 1. For a finite indexing set I the sum and the product are the same. That is if $|I| < \infty$ then $\prod_{\alpha\in I} G_\alpha = \bigoplus_{\alpha\in I} G_\alpha$.

2.2. **Module.** A module is essentially a generalization of a vector space. Recall a vector space is an abelian group endowed with multiplication from a *field* k, often taken to be $\mathbb{C}$ or $\mathbb{R}$. Throughout I will assume $R = (R, +, \cdot_R)$ is a commutative ring with unity 1.

Definition 3. An R-module M is an abelian group equipped with a "scalar" multiplication map from R, that is a map

$$\cdot : R \times M \to M$$
$$(r, m) \to r \cdot m$$

such that for all $m, n \in M$ and $r \in R$ we have

- $r \cdot (m + n) = rm + rn$
- $(r + s) \cdot m = rm + sm$
- $r \cdot (s \cdot (m)) = (r \cdot_R s) \cdot M$
- $1_R \cdot m = m$

I will use $\cdot$ to denote module multiplication between elements of R and M and $\cdot_R$ to denote multiplication in R. (Until I get tired and drop the subscript.) Moreover, If the ring R is understood or is just some arbitrary ring, I may occasionally drop the word R-module and just say module.

Remark 2. If we did not assume R is commutative then Definition 3 would technically define a *left R-module*. A right R-module would be defined analogously. The commutative assumption however allows one to not have to distinguish between the two.

Wednesday January 14:
"You always have to realize that you're constantly in the state of becoming, you know?"
-Bob Dylan

Example 3.
- Every abelian group G is a $\mathbb{Z}$-module. The multiplication rule for any $z \in \mathbb{Z}_{\geq 0}$, $z' \in \mathbb{Z}_{<0}$ and $g \in G$ is
$$z \cdot g = \underbrace{g \cdot g \cdots g}_{z-\text{times}} \text{ and } \quad z' \cdot g = \underbrace{(-g) \cdot (-g) \cdots (-g)}_{|z'|-\text{times}}.$$
- In the case when the ring R in Definition 3 happens to be a field an R module is just a vector space.
- The polynomial ring $\mathbb{C}[x]$ is a $\mathbb{C}$-module. Do you think it is also an $\mathbb{R}$-module? How about a $\mathbb{Z}$ module? Is the polynomial ring $\mathbb{Z}[x]$ also a $\mathbb{C}$-module?
- There is nothing special about one variable. In fact for any ring R, the polynomial ring $R[x_0, \ldots, x_n]$ is an R-module. Make sure you know the rule of multiplication.

2.3. **Basis of a Module and a Universal Property.** Recall that for an abelian group $(M, +)$, a subset $S \subseteq M$ is a generating set for M if we can write every element of M as a finite combination of elements of S. More formally S is a generating set if and only if for all $m \in M$
$$m = s_1^{k_1} \cdot s_2^{k_2} \cdots s_n^{k_n} = \sum_{i=1}^{n} k_i s_i \text{ such that } k_i \in \mathbb{Z}.$$
The last equality is only notational, instead of writing $\cdot$ to denote the operation of M we are using $+$ to denote this operation as this is what we will use for R-modules. Definition 4 extends the definition of a generating set to one of basis for R-modules where the set S satisfies the notion of "linear independence" as well. (This is all just extending the notion of the basis of a vector space.)

Definition 4. A basis of an R-module M is a subset $B \subseteq M$ such that:
(1) for any $m \in M$ there exists a finite subset $\{b_1, \ldots, b_n\} \subset B$ where $m = \sum_{i=1}^{n} r_i b_i$ with $r_i \in R$.
(2) For any finite subset $\{b_1, \ldots, b_n\} \subset B$ and $r_i \in R$ we have
$$\text{if } r_1 b_1 + \cdots r_n b_n = 0 \text{ then } r_1 = r_2 = \cdots = r_n = 0.$$

Caution: Unlike vector space land where every vector space has a basis, it is **not** true that every R module has a basis. Those that do are special and are called *free R modules.*

Example 4.
(1) Every ring R is a free R-module where the multiplication rule is given by $\cdot_R$ and the basis is 1_R.
(2) The set $\{1, x, x^2, x^3, \ldots\} = \{x^i\}_{i \in \mathbb{N}}$ is an R-module basis for $R[x]$.

(3) A good exercise (and perhaps a quiz problem!) would be to write an R-module basis for $R[x, y]$.

(4) The group $\mathbb{Z}^m := \underbrace{\mathbb{Z} \oplus \mathbb{Z} \cdots \oplus \mathbb{Z}}_{m}$ has a basis given by $\{e_i\}_{i=1}^m$ where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$; that is 1 appears in the i-th spot. In fact the same is true for R^m when perceived as an R-module.

Definition 5. Given any set X, the free R-module generated by X (or spanned by X) is given as

$$R\langle X\rangle = \bigoplus_{x\in X} R = \{f : X \to R \mid f(x) = 0 \text{ for all but finitely many } x\}.$$

The cardinaly of X, denoted $|X|$, is called the *rank of* $R\langle X\rangle$.

In fact one can show that X is indeed a basis of $R\langle X\rangle$. Before introducing the universal property of a basis let us define an R-module homomorphism. Recall that a group homomorphism was a map that "respected" the group operations. Intuitively, an R-module homomorphism is a map that respects the group operations as well as the rule of multiplication by R. Here is the formal definition.

Definition 6. Let M and N be R-modules. An *R-module homomorphism* is a function $\phi : M \to N$ such that for all $m_1, m_2 \in M$ and all $r \in R$:

- $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$
- $\phi(r \cdot_M m_1) = r \cdot_N \phi(m_1)$ (the left hand side uses the rule of multiplication for M and the right hand side the rule of multiplication for N.)

The intuition behind Theorem 2.1, which I won't prove, is essentially a generalization of the following fact: when dealing with vector spaces it is enough to define a map on a basis only. The map then uniquely extends to the rest of the vector space. The same is true for R-modules.

Theorem 2.1 (The Universal Property of an R-module basis)**.** *An R-module $(M, +, \cdot)$ is free with basis X if and only if given any other R-module $(N, +_N, \cdot_N)$ and any function $\hat{h} : X \to N$, there exists a unique R-module homomorphism h, extending $\hat{h}$ to all of M, that is*

$$h : M \to N, \quad \text{and } h(x) = \hat{h}(x) \text{ for all } x \in X.$$

Theorem 2.2. *Any R-module M is a quotient of a free R-module.*

Proof. Let X be a generating set for M. For example one could choose all of M as a generating set for M. While this is an inefficient choice it shows that M has at least one generating set thus our choice is not bogus. Note that $X \subseteq M$ and by Theorem 2.1, the inclusion map $i : X \to M$, which is defined as $i(x) = x$, uniquely extends to an R-module homomorphism $h : R\langle X\rangle \to M$. Note that $R\langle X\rangle$ is a free R-module (Definition 5). Moreover the map h is onto; for $m \in M$ the definition of generation implies $m = \sum_{i=1}^{n} r_i x_i$ where $x_i \in X$

and $r_i \in R$ for all i. The first isomorphism theorem* implies

$$\frac{R\langle X\rangle}{\ker(h)} \cong M \qquad \square$$

*All three isomorphism theorems for groups also hold for R-modules. While I won't prove this the proof is essentially identical to the group case.

Next we are going to focus on $\mathbb{Z}$-modules, however most of these results hold true for R-modules where R is a Principal Ideal Domain (PID).

Friday January 16:
"I believe, every day, you should have at least one exquisite moment."

-Audrey Hepburn.

Theorem 2.3. *For any subgroup $G \subseteq \mathbb{Z}^m$ there exists a basis $\{b_1, \ldots, b_m\}$ of $\mathbb{Z}^m$ such that $G = \mathbb{Z}\langle k_1 b_1, \ldots, k_r b_r\rangle$ where $r \leq m$, $k_i \in \mathbb{Z}$ for all i and $k_i \mid k_{i+1}$, that is G is a free $\mathbb{Z}$-module generated by $\{k_1 b_1, \ldots, k_r b_r\}$.*

Corollary 1. *For any finitely generated $\mathbb{Z}$-module G with a basis of size m, there exists $k_1, \ldots, k_r \in \mathbb{Z}$ with $k_i \mid k_{i+1}$ such that*

$$G \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m-r} \oplus \underbrace{\frac{\mathbb{Z}}{k_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{k_r\mathbb{Z}}}_{r\,(\text{torsion part})}. \tag{2.1}$$

Moreover the choice of k_i's depends only on G.

In expression Equation (2.1):

- the integer $m - r$ is called the free rank or the Betti number of G,
- the integers k_i are called the invariant factors of G,
- and the form in which G is decomposed is called the *invariant factor decomposition of G.*

One can massage Equation (2.1) into a slightly different shape by writing a prime decomposition of each k_i and using the isomorphism

$$\frac{\mathbb{Z}}{rZ} \oplus \frac{\mathbb{Z}}{s\mathbb{Z}} \cong \frac{\mathbb{Z}}{sr\mathbb{Z}} \quad (\text{where } \gcd(r, s) = 1)$$

to write Equation (2.1) as

$$G \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{m-r} \oplus \left(\frac{\mathbb{Z}}{p_1\mathbb{Z}}\right)^{a_1} \oplus \cdots \oplus \left(\frac{\mathbb{Z}}{p_t\mathbb{Z}}\right)^{p_t}. \tag{2.2}$$

where a_i is the sum total of the number of times the prime p_i appears in the prime decomposition of all the a_i's. Equation (2.2) is called the *elementary divisor decomposition of G.* You can find all of this in [Dum99, Section 5.2].

Our main interest in basis and decompositions 2.1,2.2 is the ability to express maps between $\mathbb{Z}$-modules in terms of matrices. It enables us to do "linear algebra" over $\mathbb{Z}$. If I forget to mention it, our morphisms going forward will be considered as module homomorphisms as in Definition 6.

Upon fixing a choice of basis for $\mathbb{Z}^n$ and $\mathbb{Z}^m$ any homomorphism

$$f : \mathbb{Z}^n \to \mathbb{Z}^m$$

can be identified with an $m \times n$ matrix with $\mathbb{Z}$ entries

$$A_f = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

such that for any $x \in \mathbb{Z}^n$, where x is written in terms of the basis i.e. $x = \sum_{i=1}^{n} x_i e_i$, we have

$$f(x) = A_f \cdot \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Our goal is to reduce a matrix with integer entries into something analogous to "reduced echelon form" in linear algebra. To do so certain operations on matrices with entries in $\mathbb{Z}$ are allowed, namely:

- exchanging rows i and k,
- multiplying a row by -1,
- replacing row i with (row i)$+q$(row k) where $q \in \mathbb{Z}$ and $k \neq i$.

Moreover, the same kind of operations can be performed on the columns. These operations are allowed because they essentially amount to a change of basis of the domain (row operations) and a change of basis of the codomain (column operations).

Theorem 2.4 (Smith Normal Form)**.** *Let $G = \mathbb{Z}^n$ and $G' = \mathbb{Z}^m$ i.e. free abelian groups of ranks n and m and $f : G \to G'$ be a homomorphism. Then there exists basis of G and G' such that*

$$A_f = \left[\begin{array}{cccc|c} b_1 & 0 & \dots & 0 & \\ 0 & b_2 & \dots & 0 & \\ & & \ddots & & \mathbf{0} \\ 0 & 0 & \dots & b_l & \\ \hline \multicolumn{4}{c|}{\mathbf{0}} & \mathbf{0} \end{array}\right] \tag{2.3}$$

where $b_i \geq 1$ and $b_i \mid b_{i+1}$ for all $1 \leq i \leq l$. The matrix A_f is called the Smith normal form of f.

If one is willing to accept the fact that any subgroup of a free abelian group is free i.e. if $G \subseteq \mathbb{Z}^m$ then G is free, then Theorem 2.3 becomes a corollary of Theorem 2.4. To make this more precise, since $G \subseteq \mathbb{Z}^m$ which makes G free, then by Theorem 2.4 one can find basis for G and $\mathbb{Z}^m$ to achieve the smith normal form for the $i : G \hookrightarrow \mathbb{Z}^m$, the inclusion map. Using the first isomorphism theorem one concludes Theorem 2.3.

2.4. **Reduction Algorithm.** Next we are going to describe the two-step reduction algorithm which is a procedure for putting a matrix into its Smith normal form. This procedure produces an algorithmic proof of Theorem 2.4.

(1) $A = (a_{ij})$ is a matrix with integer entries not all of which are zero.
(2) $\alpha(A) = \min\{|a_{ij}|\}$, the smallest value of the absolute values of a_{ij}'s.

(3) An element a_{ij} is called a *minimal entry* of A if $\alpha(A) = a_{ij}$.

The two steps of the reduction algorithm for a matrix consist of reducing α (step 1) and reducing the dimensions of the matrix involved (step 2).

Step 1: Use elementary row operations to decrease the value of the function α. (remember we can only work with entries from $\mathbb{Z}$.)

Lemma 1. *If $\alpha(A)$ fails to divide some entry of A then it is possible to decrease the value of $\alpha(A)$ by applying elementary row operations to A; and conversely.*

Proof. The proof is not complicated or hard but rather just a book keeping proof. See [Mun84, page. 56] □

Example 5. Let $A = \begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix}$ and note $\alpha(A) = 2$ which does not divide every entry of A.

$$\begin{bmatrix} 2 & -7 \\ 3 & 5 \end{bmatrix} \overset{R_2=R_2-R_1}{\to} \begin{bmatrix} 2 & -7 \\ 1 & 12 \end{bmatrix} \overset{R_2\leftrightarrow R_1}{\to} \begin{bmatrix} 1 & 12 \\ 2 & -7 \end{bmatrix} \overset{R_2=-2R_1+R_2}{\to} \begin{bmatrix} 1 & 12 \\ 0 & -31 \end{bmatrix}$$
$$\overset{C_2=-12C_1+C_2}{\to} \begin{bmatrix} 1 & 0 \\ 0 & -31 \end{bmatrix} \overset{C_2=-C_2}{\to} \begin{bmatrix} 1 & 0 \\ 0 & 31 \end{bmatrix}$$

Step 2: After step 1 we have achieved a matrix where $\alpha(A)$ divides every entry of A. Perform row/column swaps to position the entry a_{ij}, where $\alpha(A) = a_{ij}$, to the a_{11} position of the matrix (upper left corner) and make it positive. Since a_{11} divides everything in its row and column we can use appropriate row/column operations to make the first row and column zero. Our matrix now looks something like:

$$A = \left[\begin{array}{c|c} a_{11} & \mathbf{0} \\ \hline \mathbf{0} & A' \end{array}\right]$$

We then repeat steps 1 and 2 for A' which is an $(n-1) \times (n-1)$ matrix.

Example 6. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}$. Our first step is to make sure $\alpha(A)$ is minimized; since it does not divide every entry of A Lemma 1 implies we can minimize $\alpha(A)$.

$$\begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix} \to \begin{bmatrix} 3 & 3 \\ 0 & 10 \end{bmatrix} \to \begin{bmatrix} 3 & 3 \\ -9 & 1 \end{bmatrix} \to \begin{bmatrix} -9 & 1 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -9 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 3 & 30 \end{bmatrix}$$

The reduction above can be interpreted as showing the cokernels of the maps ϕ and ϕ' are isomorphic.

$$\phi : \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 3 & 0 \\ 0 & 10 \end{bmatrix}} \mathbb{Z}^2; \quad \text{image}(\phi) = 3\mathbb{Z} \oplus 10\mathbb{Z}; \quad \text{cokernel}(\phi) = \frac{\mathbb{Z}^2}{\mathbb{Z}_3 \oplus \mathbb{Z}_{10}} \cong \frac{\mathbb{Z}}{3\mathbb{Z}} \oplus \frac{\mathbb{Z}}{10\mathbb{Z}}$$

$$\phi' : \mathbb{Z}^2 \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 30 \end{bmatrix}} \mathbb{Z}^2; \quad \text{image}(\phi) = \mathbb{Z} \oplus 30\mathbb{Z}; \quad \text{cokernel}(\phi) = \frac{\mathbb{Z}^2}{\mathbb{Z} \oplus 30\mathbb{Z}} \cong \frac{\mathbb{Z}}{30\mathbb{Z}}$$

Since ϕ' is a result of ϕ by "legal" matrix operations their cokernels should be isomorphic which is indeed the case.

3. Chain Complexes and Homology

Wednesday January 21:
"Ring the bells that still can ring. Forget your perfect offering. There is a crack in everything. That's how the light gets in."

-Leonard Cohen.

3.1. **Definitions and Background.** We will use the concepts developed in Section 2, namely doing linear algebra over $\mathbb{Z}$, to compute *homology of chain complexes*.

Definition 7. Let $\{C_p\}_{p\in\mathbb{Z}}$ be a sequence of abelian groups along with a sequence of homomorphisms

$$\delta_p : C_p \to C_{p-1}$$

such that $\delta_p \circ \delta_{p+1} = 0$. A running assumption for us is that $C_p = 0$ (the 0-group) when $p < 0$ (although this is not necessary.) Such data of groups and maps can be represented as

$$\cdots C_{p+1} \xrightarrow{\delta_{p+1}} C_p \xrightarrow{\delta_p} C_{p-1} \xrightarrow{\delta_{p-1}} C_{p-2} \cdots C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{0} 0.$$

We call this data of groups and maps a chain complex, denoted by $C_* = (C_p, \delta_p)_{p\in\mathbb{Z}}$. The maps δ_p are called boundary maps. The subscript p is called the *grading of* C_*.

Remark 3. Given a chain complex C_*, if there is an $m \geq 0$ such $C_p = 0$ for all $p > m$ we define *the dimension of* C_* to be m; denoted by $\dim(C_*)$. If $m < \infty$ we say that C_* has finite dimension. Occasionally this is called "the length of C_*".

Here are some examples of complexes.

Example 7. Here is a somewhat trivial example. Take 4 copies of $\mathbb{Z}$ and all 0 maps.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0.$$

Example 8. Let k be a field of characteristic 0 and $R = \frac{k[x,y]}{(xy)}$. In your homework you will show that

$$0 \longrightarrow R \xrightarrow{\cdot y} R \xrightarrow{\cdot x} R \longrightarrow 0,$$

is in fact a chain complex

Example 9. Show that

$$0 \longrightarrow \mathbb{Z}_8 \xrightarrow{\cdot 4} \mathbb{Z}_8 \xrightarrow{\cdot 4} \mathbb{Z}_4 \longrightarrow 0,$$

is a chain complex.

Observation 3.1. The condition $\delta_p \circ \delta_{p+1} = 0$ in Definition 7 shows that

$$\mathrm{Im}(\delta_{p+1}) \subset \ker(\delta_p) \subset C_p.$$

This observation leads to Definition 8.

Definition 8. Given a chain complex C_*, the *k-th homology group of* C_* is defined by

$$\frac{\ker(\delta_k)}{\mathrm{Im}(\delta_{k+1})},$$

and is denoted by any of the following equivalent notations

$$H_k = H_k(C_*, \delta_*) = H_k(C_*) := \frac{\ker(\delta_k)}{\mathrm{Im}(\delta_{k+1})}.$$

The direct sum of homology groups is denoted by $H_*(C_*)$ and is given as

$$H_*(C_*) = \bigoplus_{k\geq 0} H_k(C_*).$$

Notation 1. For a given chain complex C_*:

- the module C_p is said to be in *homological degree* p,
- $C_p :=$ is the *group of* p*-chains*,
- $Z_p := \ker(\delta_p)$ is the *subgroup of* p*-cycles*,
- $B_p := \mathrm{Im}(\delta_{p+1})$ is called a *subgroup of* p*-boundaries*,
- p-boundaries $\subseteq$ p-cycles $\subseteq$ p-chains.

Therefore $H_p = \dfrac{Z_p}{B_p}$ and the slogan for homology is "cycles mod boundaries". A typical element of H_p looks like $[\alpha] = \alpha + B_p$ where $\alpha \in Z_p$. When clear I'll avoid using the $[\alpha]$ notation and just write α to represent an entire equivalence class.

Recall from Definition 3 that any R-module is an abelian group equipped with multiplication from R. Therefore it is sensible to consider a sequence of R-modules in Definition 7. While there is a beautiful and active area of research that deals with computing homologies of R-modules, broadly called homological (commutative) algebra, our focus will be on chain complexes $C_* = (C_p, \delta_p)$ that are collections of "finitely generated free $\mathbb{Z}$ modules" i.e. $C_p \cong \mathbb{Z}^{k_p}$ where $k_p \geq 0$.

Example 10. Let C_* be a chain complex where $C_0 = \mathbb{Z}$ and $C_i = \mathbb{Z}^{2i}$ for $i \geq 1$. I won't specify what the maps are but the complex will look like

$$\cdots \longrightarrow \underbrace{\mathbb{Z}^8}_{C_4} \xrightarrow{\delta_4} \underbrace{\mathbb{Z}^6}_{C_3} \xrightarrow{\delta_3} \underbrace{\mathbb{Z}^4}_{C_2} \xrightarrow{\delta_2} \underbrace{\mathbb{Z}^2}_{C_1} \xrightarrow{\delta_1} \underbrace{\mathbb{Z}}_{C_0} \xrightarrow{\delta_0} 0$$

Clearly δ_0 is the zero map. The key point is the choice of C_p allows one to use Theorem 2.4. Upon fixing a basis for each C_i one gets that δ_1 can be represented by a 2×1 matrix and for $i \geq 2$, each δ_i can be represented by a $2i \times 2(i-1)$ matrix.

Our main objective is to answer two questions about the homology groups of chain complexes of finitely generated free $\mathbb{Z}$-modules:

- Can we compute these homology groups?
- What is the Betti number and the Torsion part of each group?

For the remainder of section 3 I will assume that all chain complexes consist of finitely generated free $\mathbb{Z}$-modules.

Recall a free $\mathbb{Z}$-module meant that there are no relations among the generators. Consider Notation 1 and note that both Z_p and B_p are free subgroups of C_p; because a subgroup of a free group is free. However, the p-th homology group $H_p \cong \dfrac{Z_p}{B_p}$ is only finitely generated and not necessarily free. Corollary 1 implies that

$$H_p \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\mathbb{Z}^{\beta_p}} \oplus \underbrace{\frac{\mathbb{Z}}{k_1\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{k_r\mathbb{Z}}}_{T}. \tag{3.1}$$

where $\mathrm{rank}(H_p) = \beta_p$ and is called *the* p*-th Betti number* of the complex C_* and T stands for Torsion.

3.2. **Computing Betti numbers and Torsions of H_p.** Theorem 3.2 gives a decomposition of C_p from which we will extract the needed information in Corollary 2.

Theorem 3.2 (Standard Bases for Free Chain Complexes)**.** [Mun84, Theorem 11.4] *Let C_* be a chain complex of finitely generated free $\mathbb{Z}$-modules. For each p there exists subgroups U_p, V_p and W_p such that*

$$C_p = U_p \oplus \underbrace{V_p \oplus W_p}_{Z_p}$$

where $\delta_p(U_p) \subseteq W_{p-1}$ and $\delta_p(V_p) = \delta_p(W_p) = 0$. Moreover, W_p is a uniquely determined subgroup of C_p while V_p and U_p are not. Furthermore, there exists basis for U_p and W_{p-1} relative to which the map $\delta_p : U_p \to W_{p-1}$ has the form

$$\begin{bmatrix} b_1 & 0 & \dots & 0 \\ 0 & b_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & b_l \end{bmatrix} \tag{3.2}$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid b_3 \mid \cdots \mid b_l$.

Friday January 23:
"In order to control myself, I must first accept myself, by going with and not against my nature."

-Bruce Lee

Proof. In step 1 we construct W_p and V_p and show each is a summand of C_p. The summand U_p is constructed in step 3.

Step 1: Fix p and suppose $C_p \cong \mathbb{Z}^n$ and $C_{p-1} \cong \mathbb{Z}^m$. Then part of C_* looks like

$$\cdots C_{p+1} \xrightarrow{\delta_{p+1}} \underbrace{C_p}_{\cong \mathbb{Z}^n} \xrightarrow{\delta_p} \underbrace{C_{p-1}}_{\cong \mathbb{Z}^m} \longrightarrow \cdots$$

Recall $Z_p = \ker(\delta_p)$ and $B_p = \operatorname{Im}(\delta_{p+1})$. Define W_p as the set of elements c_p of C_p where some multiple of c_p is in B_p, that is

$$W_p = \{c_p \in C_p \mid m \cdot c_p \in B_p \text{ for some } m \in \mathbb{Z}_{>0}\}.$$

In your homework you will justify that W_p is indeed a subgroup of C_p as well as the $(*)$ inclusions

$$B_p \overset{*}{\subset} W_p \overset{*}{\subset} Z_p \subset C_p$$

The subgroup W_p is called the *weak boundaries*. Using the definition of $H_p = \frac{Z_p}{B_p}$ and Equation (3.1) we get the projections

$$Z_p \overset{\pi_1}{\twoheadrightarrow} H_p \overset{\pi_2}{\twoheadrightarrow} \frac{H_p}{T_p} \Rightarrow Z_p \overset{\pi}{\twoheadrightarrow} \frac{H_p}{T_p} \text{ where } \pi = \pi_2 \circ \pi_1.$$

You will show, as part of your homework, that $\ker(\pi) = W_p$. Thus the first isomorphism theorem implies $\dfrac{Z_p}{W_p} \cong \dfrac{H_p}{T_p}$. Since $\dfrac{H_p}{T_p}$ is finitely generated and torsion free group then it is a free group thus $\dfrac{Z_p}{W_p}$ is a free group. Fix $c_1 + W_p, \dots, c_k + W_p$ to be a basis of $\dfrac{Z_p}{W_p}$ and fix a

representative $c_1, \ldots, c_k$ of each class. Moreover, W_p is a free group (why?), so fix $d_1, \ldots, d_l$ be a basis of W_p. Then one can show that $c_1, \ldots, c_k, d_1, \ldots, d_l$ is a basis of Z_p thus

$$Z_p = \underbrace{V_p}_{\langle c_1,\ldots,c_k\rangle} \oplus \underbrace{W_p}_{\langle d_1,\ldots,d_l\rangle}. \tag{3.3}$$

Notice that W_p is uniquely determined because the torsion part of H_p is uniquely determined by Corollary 1.

Step 2: By Theorem 2.4 we can choose basis $e_1, \ldots, e_l, e_{l+1}, \ldots, e_n$ and $e'_1, \ldots, e'_l, e'_{l+1}, \ldots, e'_m$ for C_p and C_{p-1} with respect to which δ_p looks like:

$$\begin{array}{c} \\ e'_1 \\ e'_2 \\ \vdots \\ e'_l \\ e'_{l+1} \\ \vdots \\ e'_m \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & \ldots & e_l & e_{l+1} & \ldots & e_n \end{array} \\ \left[\begin{array}{cccc|ccc} b_1 & 0 & \ldots & 0 & & & \\ 0 & b_2 & \ldots & 0 & 0 & \ldots & 0 \\ & & \ddots & & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & b_l & 0 & \ldots & 0 \\ \hline 0 & \ldots & 0 & & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & & 0 & \ldots & 0 \end{array}\right] \end{array} \tag{3.4}$$

where $b_i \geq 1$ and $b_1 \mid b_2 \mid b_3 \mid \cdots \mid b_l$. Next, we will prove the following:

(1) $e_{l+1}, \ldots, e_n$ is a basis for Z_p,
(2) $e'_1, \ldots, e'_l$ is a basis for W_{p-1},
(3) $b_1 e'_1, \ldots, b_l e'_l$ is a basis for $B_{p-1} = \mathrm{Im}(\delta_p)$.

Let c_p be a general element of C_p to which we will apply to boundary map δ_p.

$$c_p = \sum_{i=1}^{n} a_i e_i \Rightarrow \delta_p(c_p) \underset{3.4}{=} \sum_{i=1}^{l} a_i b_i e'_i$$

Note that $c_p \in Z_p = \ker(\delta_p)$ if and only if $\delta_p(c_p) = 0$ if and only if $a_i = 0$ for all $1 \leq i \leq l$ since all b_i's are non-zero and e'_i's are linearly independent. Thus $Z_p = \mathbb{Z}\langle e_{l+1}, \ldots, e_n\rangle$ which proves (1). By Equation (3.4) it is clear that $b_1 e'_1, \ldots, b_l e'_l$ is a generating set for B_{p-1}. Since none of the b_i's are zero and e'_is are independent we get that $b_1 e'_1, \ldots, b_l e'_l$ is independent and (3) follows. To see (2) note for each $1 \leq i \leq l$, the element e'_i belongs to W_{p-1} because $b_i \cdot e'_i$ is inB_{p-1} thus $\mathbb{Z} \cdot \langle e'_1, \ldots, e'_l\rangle \subset W_{p-1}$. To see the other inclusion, let $c_{p-1} \in C_{p-1}$ i.e. $c_{p-1} = \sum_{i=1}^{m} d_i e'_i$ such that $c_{p-1} \in W_{p-1}$.

$$\begin{aligned} & \lambda \cdot c_{p-1} \in B_{p-1} \text{ for } \lambda \neq 0 \\ \Rightarrow & \lambda \cdot c_{p-1} = \delta_p(c_p) \text{ for some } c_p \in C_p. \\ \Rightarrow & \lambda \cdot c_{p-1} \underset{\text{part 3}}{=} \sum_{i=1}^{l} a_i b_i e'_i. \\ \Rightarrow & \sum_{i=1}^{m} \lambda d_i e'_i = \sum_{i=1}^{l} a_i b_i e'_i \Rightarrow d_i = 0 \text{ for } l < i \leq m \\ \Rightarrow & c_{p-1} = \sum_{i=1}^{l} d_i e'_i \in \mathbb{Z}\langle e'_1, \ldots, e'_l\rangle \end{aligned}$$

Finally, since e_i''s are independent, part (2) follows.

Step 3: Using our choice of basis in step 2 let U_p be the group spanned by $e_1, \dots, e_l$ therefore

$$C_p = U_p \oplus Z_p = U_p \oplus \underbrace{V_p \oplus W_p}_{Z_p} \tag{3.5}$$

where the second equality follows from Equation (3.3). By definition of Z_p we have $\delta_p(V_p) = \delta_p(W_p) = 0$ and Equation (3.2) follows from step 2 (ultimately it's from Theorem 2.2.) □

Monday January 26:

"What keeps my heart awake is colorful silence."

-Claude Monet.

Corollary 2 is the same as [Mun84, Theorem 11.5]. It requires a fixed choice of basis (and orientation). I we won't get into that, just note that I am not telling you the entire truth.

Corollary 2 (Betti Number)**.** *The decomposition of the p-th homology group of C_* is given by*

$$H_p \cong \mathbb{Z}^{\beta_p} \oplus \mathbb{Z}/r_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r_t\mathbb{Z} \tag{3.6}$$

where $\beta_p = null\ (\delta_p) - rank\ (\delta_{p+1})$ and $r_1, \dots, r_t$ are the entries in the Smith normal form of the matrix $\delta_{p+1} : C_{p+1} \to C_p$.

Proof. Let M_p represent the matrix in Equation (3.4). Rewriting our three observations about M_p we get

(1) the rank of Z_p is given by the number of zero columns of M_p,
(2) the rank of W_{p-1} is given by the number of non-zero rows of M_p,
(3) $\dfrac{W_{p-1}}{B_{p-1}} \cong \mathbb{Z}/b_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/b_l\mathbb{Z}$.

We will not directly use 1 and 2 but I wanted to have this interpretation written somewhere. Using Equation (3.5) we will write a decomposition of $H_p = Z_p/B_p$ and keep in mind that $\beta_p = \mathrm{Im}(\delta_{p-1})$.

$$H_p = \frac{Z_p}{B_p} \underset{3.5}{\cong} \frac{V_p \oplus W_p}{0 \oplus B_p} \cong V_p \oplus \frac{W_p}{B_p} \underset{3.3}{\cong} \frac{Z_p}{W_p} \oplus \frac{W_p}{B_p}$$

Notice that $\dfrac{Z_p}{W_p}$ is free (because it is finitely generated and torsion free) therefore $\dfrac{Z_p}{W_p} \cong \mathbb{Z}^q$. Since $\dfrac{W_p}{B_p}$ is the torsion part and β_p is, by definition, the rank of the free part of H_p we get $q = \beta_p$. Finally we compute β_p:

$$\beta_p = \mathrm{rank}(Z_p) - \mathrm{rank}(W_p) = \mathrm{null}(\delta_p) - \mathrm{rank}(\delta_{p+1}) \tag{3.7}$$

and recall $\beta_p = \mathrm{Im}(\delta_{p+1})$. Let

$$\left[\begin{array}{cccc|ccc} r_1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 & \vdots & \ddots & \vdots \\ & & \ddots & & 0 & \dots & 0 \\ 0 & 0 & \dots & r_t & & & \\ \hline 0 & \dots & 0 & & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & & 0 & \dots & 0 \end{array}\right]$$

be the Smith normal form of $\delta_{p+1} : C_{p+1} \to C_p$. Item (3) above implies

$$\frac{W_p}{B_p} \cong \mathbb{Z}/r_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/r_t\mathbb{Z}.$$

Finally, putting Equation (3.7) and Section 3.2 yields Equation (3.6). □

Let us put all of this hard work into computing an actual example.

Example 11. Consider the chain complex

$$0 \xrightarrow{\delta_2} \underbrace{\mathbb{Z}^3}_{C_1} \xrightarrow{\left(\begin{smallmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)} \underbrace{\mathbb{Z}^3}_{C_0} \xrightarrow{\delta_0} 0.$$

Recall the rank of a matrix is the number of linearly independent columns. In fact $\mathrm{rank}(\delta_1) = 2$ because column(1) $= -1\cdot$column(2)+column(3) and one can show columns 2 and 3 are independent. Moreover

$$\begin{bmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.8}$$

Case 1: $p = 0$. By Equation (3.7) we have

$$\beta_0 = \mathrm{null}(\delta_0) - \mathrm{rank}(\delta_1) = 3 - 2 = 1$$

Be careful that $\beta_0 = 1$ only tells us that H_0 *contains* a copy of $\mathbb{Z}$ but there may still be a torsion part. In this example however $H_0 \cong \mathbb{Z}$. We did not really have to compute β_0 as above because Equation (3.8) shows $B_0 = \mathrm{Im}(\delta_1) = \mathbb{Z}^2$ therefore

$$H_0 = \frac{Z_0}{B_0} \cong \frac{\mathbb{Z}^3}{\mathbb{Z}^2} \cong \mathbb{Z}.$$

Case 2: $p = 1$. From Equation (3.8) we see that $Z_1 = \ker(\delta_1) \cong \mathbb{Z}$. Further $B_1 = \mathrm{Im}(\delta_2) = 0$ therefore $H_1 = \dfrac{Z_1}{B_1} \cong \mathbb{Z}$.

$$H_p \cong \begin{cases} \mathbb{Z} \ ; \ p = 0, 1 \\ 0 \ ; \ \text{else} \end{cases} \qquad \text{and} \qquad \underbrace{H_*(C_*) = \bigoplus_p H_p \cong \underbrace{\mathbb{Z}}_{\deg 0} \oplus \underbrace{\mathbb{Z}}_{\deg 1}}_{\text{total homology of } C_*}.$$

4. Simplicial Complexes

Wednesday January 28:
"The day you catch an idea you fall in love with, even a small one, is a beautiful day."
-David Lynch.

4.1. ***k*-simplex.** So far every chain complex you have seen has been pulled out of thin air. Our real interest in computing homology is for simplicial complexes which arise from collections of k-simplices and which we are now ready to define, picturially first, then a formal definition.

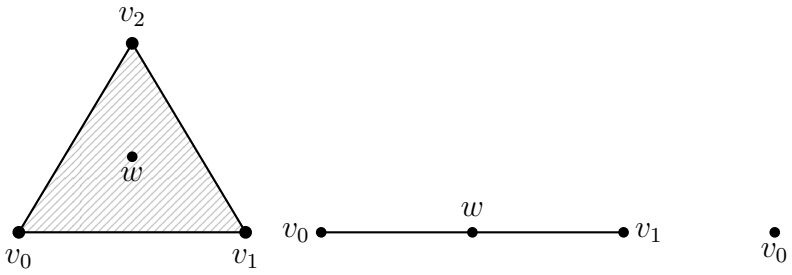


Figure 1. From left to right: 2-simplex, 1-simplex and a 0-simplex

Definition 9 (k-simplex in $\mathbb{R}^N$)**.** Fix N to be large enough. Given $k+1$ ordered vectors $v_0, \dots, v_k$ in $\mathbb{R}^N$ that do not lie on a hyperplane, meaning the vectors $v_1 - v_0, \dots, v_k - v_0$ are linearly independent, the k-simplex (defined by $v_0, \dots, v_k$) is the smallest convex set in $\mathbb{R}^N$ containing $v_0, \dots, v_k$, denoted by $[v_0, \dots, v_k]$ and Δ^k. Equivalently,

$$\Delta^k = [v_0, \dots, v_k] = \bigcap_{\substack{\{v_0,\dots,v_k\} \subset C_\alpha \\ C_\alpha \subset \mathbb{R}^N \text{ is convex}}} C_\alpha = \left\{ \sum_{i=1}^{k} t_i v_i \mid \sum_{i=1}^{k} t_i = 1,\ t_i \geq 0 \text{ and } t_i \in \mathbb{R} \right\}. \tag{4.1}$$

For example one can write each simplex in Figure 1 as

$$[v_0] = \{v_0\}$$
$$[v_0, v_1] = \{w = t_0 v_0 + t_1 + v_1 \mid t_0 + t_1 = 1\}$$
$$[v_0, v_1, v_2] = \{w = t_0 v_0 + t_1 + v_1 + t_2 v_2 \mid t_0 + t_1 + t_2 = 1\}$$

The order of the vertices of a k-simplex defines an orientation of the k-simplex as follows: for any $\sigma \in S_k$, the permutation group on k elements we have

$$[v_0, \dots, v_k] = (-1)^{|\sigma|} [v_{\sigma(0)}, \dots, v_{\sigma(k)}].$$

That is a k simplex is, up to a sign, the same the k-simplex attained by permuting its vertices. Therefore the data of a k-simplex includes the vertices and their order i.e.

$$\Delta^k = [v_0, \dots, v_k] := (\{v_0, \dots, v_k\}, \leq)$$

Moreover, any simplex Δ^k inherits the subspace topology from $\mathbb{R}^N$.

Example 12. [Orientations of a 2-simplex] All the two simplicies $[v_0, v_1, v_2]$, $[v_1, v_2, v_0]$ and $[v_2, v_0, v_1]$ have the same counter-clockwise orientation. Moreover the 2 simplicies $[v_0, v_2, v_1]$, $[v_2, v_1, v_0]$ and $[v_1, v_0, v_2]$ have the same clockwise orientation.

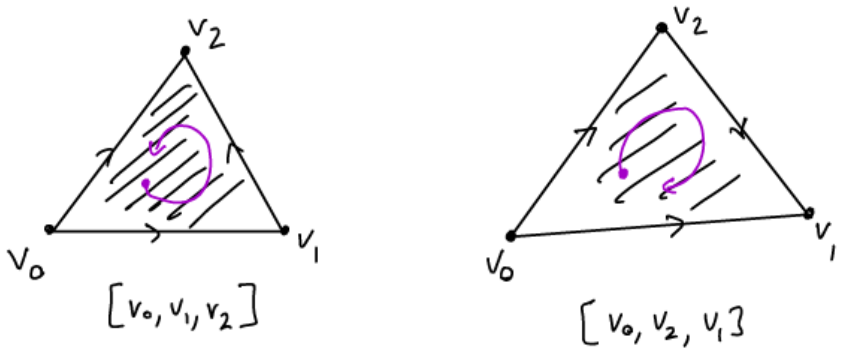


FIGURE 2. From left to right: 2-simplex with counter-clockwise orientation $[v_0, v_1, v_2]$ and a 2-simplex with a clockwise orientation $[v_0, v_2, v_1]$

Remark 4. Example 12 generalizes to a k-simplex. Given a k-simplex Δ^k there are two possibilities for the orientation of Δ^k. Moreover one can check that the orientation of a k-simplex is an equivalence relation on the set of all k-simplicies with exactly two equivalence classes. But we can't quiet use the terms "clockwise" or "counter clockwise" to describe them. If two k-simplicies differ by an even permutation they are in the same equivalence class. Therefore one way to get a simplex with a different orientation is to switch two vertices (when $k \geq 2$). A good example of this is demonstrated on [Mun84, p. 26].

Observation 4.1.
Given two simplicies $[v_0, \ldots, v_k]$ and $[w_0, \ldots, w_k]$ in $\mathbb{R}^{k+1}$, there is a canonical linear homomorphism

$$A : \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$$

such that $A([v_0, \ldots, v_k]) = [w_0, \ldots, w_k]$ where $A(v_i) = w_i$ for all i. This shows Δ^k is uniquely determined up to a linear homomorphism. Moreover, changing the order of vertices in $\Delta^k = [v_0, \ldots, v_k]$ to $\Delta^{k'} = [v_{\sigma(0)} \ldots, v_{\sigma(k)}]$ an automorphism of $\mathbb{R}^{k+1}$ where $\text{sign}(\det(A))=(-1)^{|\sigma|}$.

Definition 10. The standard k-simplex is the k-simplex given by standard basis vectors e_i that is $\Delta^k = [e_0, \ldots, e_k]$.

Example 13. Below is a drawing of the standard simplicies $\Delta^1 \subseteq \mathbb{R}^2$ and $\Delta^2 \subseteq \mathbb{R}^3$.

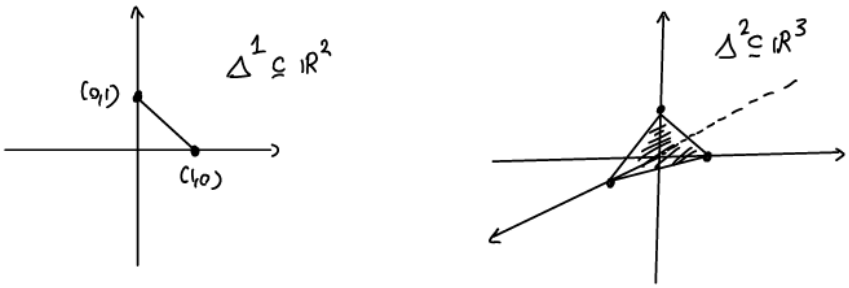


FIGURE 3. Standard simplicies $\Delta^1 \subseteq \mathbb{R}^2$ and $\Delta^2 \subseteq \mathbb{R}^3$

Definition 11 (face)**.** A face of a simplex $[v_0, \ldots, v_k]$ is a subsimplex whose vertex set is a non-empty subset of the vertices $\{v_0, \ldots, v_k\}$ with the orientation inherited from the larger simplex.

For example in Figure 1, the 2-simplex has three 0-dimensional faces, namely the vertices, three 1-dimensional faces namely the edges and one 2-dimensional face, namely the entire simplex itself. Note the vertex set of a face need not be a *proper* subset of the vertex set therefore the entire simplex is always a face of itself.

Example 14. Let $\Delta = [v_0, \dots, v_k]$ be a simplex. A subsimplex of Δ is the simplex generated by a subset of the vertices of Δ; see Figure 4. The notation $\hat{v_i}$ is to denote the vertex we are skipping.

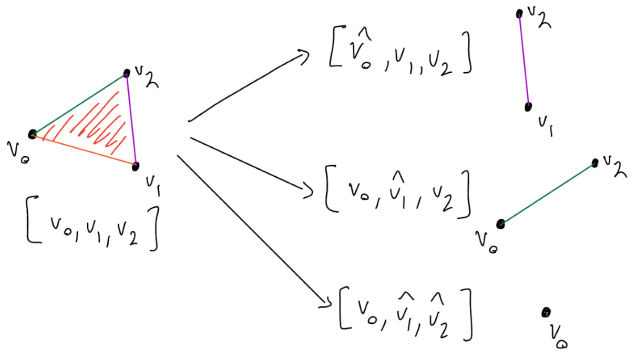


FIGURE 4. Subsimplex and removal notation

Definition 12 (Interior/Boundary)**.** The boundary of a simplex is the union of its proper faces. The interior is then defined as the complement of the boundary in the simplex.

Remark 5. You can observe this in Figure 3. The boundary of Δ^1 are just the end points which is the union of proper faces and the interior is the line segment with the end points being empty. Moreover, with this definition the interior of a zero simplex is itself; the only proper face is $\emptyset$ and the complement of $\emptyset$ is everything. In particular, note these notions of interior and boundary are different than those you defined in point set topology.

Friday January 30:
"Nothing is absolute. Everything changes, everything moves, everything revolves, everything flies and goes away."

-Frida Kahlo.

Definition 13 (Simplicial Complex)**.** Let $\{\Delta_\alpha\}$, where $\alpha \in I$ and I is a finite indexing set, be a finite family of oriented simplicies of various dimensions i.e. $\dim(\Delta_\alpha) = k_\alpha$. A finite simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$ is a union of such a family

$$\mathcal{K} = \bigcup_\alpha \Delta_\alpha$$

such that the following hold:

(1) For every α and every subsimplex (face) Δ_p of Δ_α we have $\Delta_p \in \mathcal{K}$. Said differently, if $\Delta_p \subset \Delta_\alpha$ then $\Delta_p \in \mathcal{K}$.
(2) The intersection of two simplicies in the family is a face of each simplex with the same orientation. That is $\Delta_\alpha \cap \Delta_\beta$ is a face of Δ_α and Δ_β with the same orientation.

Example 15. One can check that Figure 5 gives an example of a simplicial complex where

$$\mathcal{K} = [v_0, v_1] \cup [v_0, v_3] \cup [v_1, v_2, v_3] \cup [v_2, v_4].$$

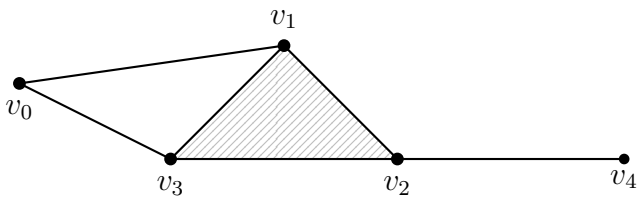


FIGURE 5. An example of a simplicial complex

Example 16. An non-example of a simplicial complex is given in Figure 6 where $\mathcal{K}' = [v_0, v_1, v_2] \cup [w_0, w_1, w_2]$. But note that $[v_0, v_1, v_2] \cap [w_0, w_1, w_2]$ is not a face of either complex.

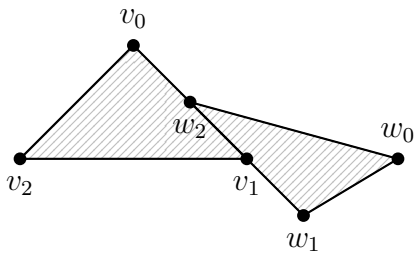


FIGURE 6. A non-example of a simplicial complex

Lemma 2. *A collection $\mathcal{K}$ of simplicies is a simplicial complex if and only if*

- *Every face of a simplex of $\mathcal{K}$ is in $\mathcal{K}$,*
- *Every pair of distinct simplicies of $\mathcal{K}$ have disjoint interiors.*

Proof. See [Mun84, Lemma 2.1, p.8] □

For example, Figure 6 demonstrates an example where the second part of Lemma 2 fails.

4.2. **Abstract Simplicial Complexes.** By now, you may have noticed that a simplicial complex can be defined combinatorially and in fact, may other areas of mathematics e.g. commutative algebra define it as such. Recall that $(\{v_0, \ldots, v_k\}, \leq)$ gives a simplicial complex with an ordering

Definition 14 (Abstract Simplicial Complex)**.** Any ordered set of vertices $(V = \{v_0, \ldots, v_k\}, \leq)$ together with a collection of subsets $\mathcal{K} = \{\Delta_\alpha\}_\alpha$, equivalently $\mathcal{K} \subseteq \mathcal{P}(V)$, the power set of V, defines a simplicial complex if and only if

- given $\Delta_\alpha \in \mathcal{K}$, every subset $\Delta_\gamma \subseteq \Delta_\alpha$ is also in $\mathcal{K}$,
- for $\Delta_\alpha, \Delta_\beta \in \mathcal{K}$, the intersection $\Delta_\alpha \cap \Delta_\beta \in \mathcal{K}$.

Example 17. Let $V = \{v_0, \ldots, v_3\}$. Observe that $\mathcal{K} = \{\emptyset, \{v_0\}, \{v_1, v_2\}, \{v_1\}, \{v_2\}, \{v_2, v_3\}, \{v_3\}\}$ defines an abstract simplicial complex.

An abstract simplicial complex $\mathcal{K}$ can be thought of as a labeling of a geometry complex. More precisely we can visualize $\mathcal{K}$ in $\mathbb{R}^{|V|}$ by choosing a basis of $\mathbb{R}^{|V|}$ that makes $\{v_0, \ldots, v_k\}$ linearly independent. And letting

$$|\mathcal{K}| = \bigcup_\alpha \Delta_\alpha \quad \text{where} \quad \Delta_\alpha = \text{conv}(\{v_\gamma \mid v_\gamma \in \Delta_\alpha\}).$$

I wish to emphasize this is *only one way* to realize an abstract simplicial complex.

Example 18. A geometric realization of $\mathcal{K}$ in Example 17 is

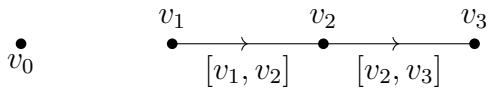


FIGURE 7. A geometric realization of Example 17

Monday February 1:
"Continue to believe in what you do and how you do it. Continue to raise your own standard among your own community of people. Never feel complacent, complacency is death... The quality of your work will open doors for you forever. Keep working until you know it's as you want it to be; not to please others, but to have it as you want it to be."

-Mira Nair.

4.3. **Simplicial Chain Complex.** Approximately 19 pages later, we have come to one of the main objects of study in this course: simplicial chain complexes. Recall from Definition 7 that one needs two pieces of data to define a chain complex: $\mathbb{Z}$-modules and maps between them. We will extract these pieces of information from a simplicial complex. The groups are defined in Definition 15 and the maps are defined in Definition 16.

Definition 15 (Simplicial Chain Groups)**.** Let $\mathcal{K} = \{\Delta_\alpha\}$ be an abstract simplicial complex and $\{\Delta_\alpha^m\}$ be the set of m-simplicies in $\mathcal{K}$. (Note this set may be empty.) The *m-th* simplicial chain group of $\mathcal{K}$ is the free abelian group generated by Δ_α^m's:

$$C_m(\mathcal{K}; \mathbb{Z}) = \bigoplus_{\Delta_\alpha^m} \mathbb{Z} = \mathbb{Z}\langle \Delta_{\alpha_1}^m, \dots, \Delta_{\alpha_r}^m \rangle \cong \mathbb{Z}^r$$

$$C_m(\mathcal{K}; \mathbb{Z}) = \left\{ \sum_{i=1}^{r} a_i \Delta_{\alpha_i}^m \right\}$$

If the set of m-th simplicies is empty for a given m then $C_m(\mathcal{K}; \mathbb{Z}) := 0$

Example 19. Continuing with Example 17 we observe that

$$C_0(\mathcal{K}, \mathbb{Z}) = \mathbb{Z}\langle v_0, v_1, v_2, v_3 \rangle \cong \mathbb{Z}^4$$
$$C_1(\mathcal{K}, \mathbb{Z}) = \mathbb{Z}\langle [v_1, v_2], [v_2, v_3] \rangle \cong \mathbb{Z}^2$$
$$C_m(\mathcal{K}; \mathbb{Z}) = 0 \text{ for any } m \geq 2.$$

Given a k-simplex $[v_0, \dots, v_k]$, let $[v_0, v_1, \dots, \hat{v_j}, \dots, V_m]$ denote the $k-1$-simplex we get by removing the j-th vertex.

Definition 16 (Simplicial Chain Maps)**.** We define the m-th boundary map as

$$\begin{aligned} \delta_m : C_m(\mathcal{K}) &\to C_{m-1}(\mathcal{K}) \\ \delta_m([v_0, \dots, v_m]) &= \sum_{j=0}^{m} (-1)^j [v_0, v_1, \dots, \hat{v_j}, \dots, v_m]. \end{aligned} \tag{4.2}$$

The map δ_m is also called the m-th boundary homomorphism.

Recall from Theorem 2.1 that defining δ_m on a basis of C_m is enough; δ_m then uniquely extends to all of C_m.

Lemma 3. *For any $m \geq 0$ we have $\mathrm{Im}(\delta_{m+1}) \subset \ker(\delta_m)$ which is equivalent to $\delta_m \circ \delta_{m+1} = 0$.*

Proof. Homework. □

Notice that Definition 15, Definition 16 and Lemma 3 describe the process of attaching a chain complex $(C_*(\mathcal{K}), \delta_*)$ to a simplicial complex $\mathcal{K}$. The *m-th homology group of the complex $\mathcal{K}$* is defined as

$$H_m(\mathcal{K}) = H_m(C_*(\mathcal{K}), \delta_*) = \frac{\ker(\delta_m)}{\mathrm{Im}(\delta_{m+1})}. \tag{4.3}$$

The m-th betti number of $\mathcal{K}$ is

$$\beta_m(\mathcal{K}) = \beta_m(C_*(\mathcal{K}), \delta_*) = \mathrm{rank}(H_m(\mathcal{K})). \tag{4.4}$$

Example 20. Consider the following simplicial complex. Let us compute simplicial chain

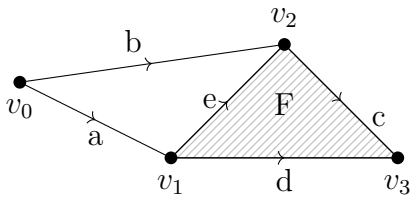


FIGURE 8. Simplicial Complex $\mathcal{K}$

groups, maps and the homology groups of $\mathcal{K} = \{[v_0, v_1], [v_0, v_2], [v_0, v_1, v_2]\}$.

$$C_0(\mathcal{K}) \cong \mathbb{Z}\langle v_0, v_1, v_2, v_3\rangle \cong \mathbb{Z}^4$$
$$C_1(\mathcal{K}) \cong \mathbb{Z}\langle \underbrace{[v_0, v_1]}_{a}, \underbrace{[v_0, v_2]}_{b}, \underbrace{[v_1, v_2]}_{e}, \underbrace{[v_1, v_3]}_{d}, \underbrace{[v_2, v_3]}_{c}\rangle \cong \mathbb{Z}^5$$
$$C_2(\mathcal{K}) \cong \mathbb{Z}\langle F\rangle \cong \mathbb{Z}$$

Thus the complex looks like

$$0 \to C_2(\mathcal{K}) \xrightarrow{\delta_2} C_1(\mathcal{K}) \xrightarrow{\delta_1} C_0(\mathcal{K}) \xrightarrow{\delta_0} 0$$

Clearly $\delta_0 = 0$. Using Equation (4.2) we get

$$\delta_1([v_0, v_1]) = [\hat{v_0}, v_1] + (-1)[v_0, \hat{v_1}] = v_1 - v_0$$
$$\delta_1([v_0, v_2]) = [\hat{v_0}, v_2] + (-1)[v_0, \hat{v_2}] = v_1 - v_2$$
$$\delta_1([v_2, v_3]) = [\hat{v_2}, v_3] + (-1)[v_2, \hat{v_3}] = v_3 - v_2$$

Notice the exponent corresponds to the order of the basis vector and not its index. For example in $[v_2, v_3]$, v_2 is the "zeroth" basis and v_3 is the "first" basis which explains the notation. You can the other two to observe

$$\delta_1 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The rows, in order, correspond to v_0, v_1, v_2, v_3 and the columns, in order, corresponds to a, b, c, d, e. We conclude $\text{rank}(\text{Im}(\delta_1)) = 3,\ \text{null}(\delta_1) = 2$

$$\delta_2(F) = \delta_2([v_1, v_2, v_3]) = [\hat{v_1}, v_2, v_3] - [v_1, \hat{v_2}, v_3] + [v_1, v_2, \hat{v_3}] = c - d + e$$

$$\Rightarrow(\delta_2) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \text{rank}(\text{Im}(\delta_2)) = 1,\ \text{null}(\delta_2) = 0$$

Putting all this information together the Homology groups compute as

$$H_0 \cong \frac{\mathbb{Z}^4}{\mathbb{Z}^3} \cong \mathbb{Z}$$

$$H_1 \cong \frac{\mathbb{Z}^2}{\mathbb{Z}} \cong \mathbb{Z}$$

$$H_2 \cong \frac{0}{0} \cong 0$$

Definition 17. Given a topological space (metric space) X a triangulation of X by a simplicial complex $\mathcal{K}$ is a homeomorphism

$$h : |\mathcal{K}| \rightarrow X$$

Example 21. Below is an example of a triangulation of S^2 with the complex $\mathcal{K} = \{[v_0, v_1, v_2], [v_1, v_2, v_3], [v_0, v_2, v_3], [v_0, v_1, v_2]\}$

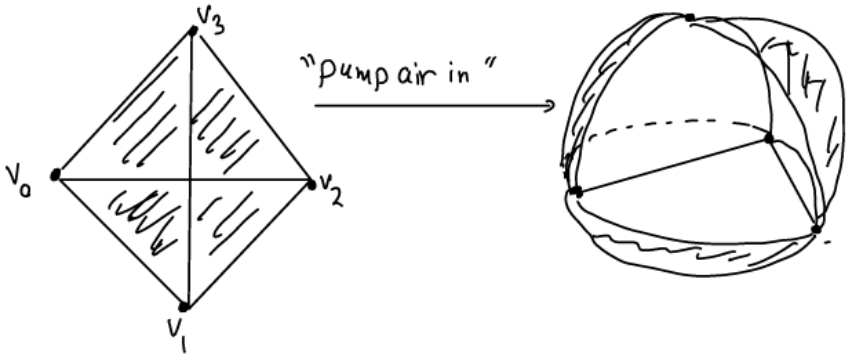


FIGURE 9. Triangulation of S^2

5. Δ-complexes

Wednesday February 4th:
"I dream of a quiet man who explains nothing and defends nothing, but only knows where the rarest wildflowers are blooming, and who goes, and finds that he is smiling not by his own will."

-Wendell Berry. (One of my top all time favorite quotes.)

A Δ-complex structure is a way of gluing simplicies together, or to a topological space, where the maps are less restrictive than simplicial complexes Definition 13. Although you may not immidiately see gluing maps in Definition 13, Figure 8 can be thought of as gluing simplicies together. So at least you see the gluing intuition pictorially.

Definition 18. If X is a topological space a Δ-complex structure on X is a collection of continuous maps $\{\sigma_\alpha\}_{\alpha\in J}$ where for $\alpha \in J$

$$\sigma_\alpha : \Delta_\alpha \longrightarrow X$$

where $\dim(\Delta_\alpha) = k_\alpha \geq 0$ and $\Delta_\alpha = [y_0^\alpha, y_1^\alpha, \ldots, y_{k_\alpha}^\alpha] \subset \mathbb{R}^N$ satisfying

(1) $\sigma_\alpha|_{\overset{\circ}{\Delta}_\alpha}$ is injective where $\overset{\circ}{\Delta}_\alpha = \mathrm{Int}(\Delta_\alpha)$ and Int stands for Interior.
(2) if $\sigma_\beta = \sigma_\alpha|_{\Delta_\beta}$ where $\Delta_\beta \subset \Delta_\alpha$ is a face, then $\sigma_\beta \in \{\sigma_\alpha\}_{\alpha\in J}$ (up to a linear homeomorphism i.e. $\sigma_\beta \circ A \in \{\sigma_\alpha\}$). In words, the restriction maps to faces, denoted here as σ_β, are also part of the collection of maps we started with.
(3) Every point $x_0 \in X$ belongs to the unique $\sigma_\beta(\overset{\circ}{\Delta}_\beta) = \mathrm{Im}(\sigma_\beta)$, meaning β is unqiue.
(4) A set u is open in X if and only if $\sigma_\alpha^{-1}(u)$ is open for all $\alpha \in J$.

Example 22. Let $X = S^1$ i.e. the circle. We put a Δ-complex structure on S^1 via the following maps:

$$\sigma_0 : \Delta_0 \to S^1 \text{ where } \Delta_0 = \{v_0\}$$
$$\sigma_0(v_0) = (1,0)$$
$$\sigma_1 : \Delta_1 \to S^1 \text{ where } \Delta_1 = [e_0, e_1]$$
$$\sigma_1(t) = \begin{cases} (1,0) & \text{if } t = e_0 \text{ or } t = e_1 \\ (\cos(2\pi t), \sin(2\pi t)) & \text{if } t \in \overset{\circ}{\Delta}_1 \end{cases}$$

To make sense of $(\cos(2\pi t), \sin(2\pi t))$, note Δ_1 can be identified with $[0,1]$ so each t, after identification with this interval, indeed corresponds to a real number.

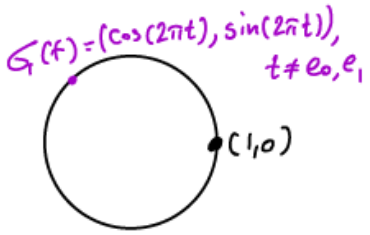


FIGURE 10. Delta Complex Structure on S^1

We can check all four axioms of Definition 18.

(1) It is clear that $\sigma_1|_{\overset{\circ}{\Delta}_1}$ is injective as values of $(\sin(t), \cos(t))$ are unique for $t \in (0,1)$.
(2) $\sigma_1|_{\{e_0\}} = \sigma_1|_{\{e_1\}} = \sigma_0$ which is indeed a map in our collection $\{\sigma_\alpha\}$.
(3) Note that the interior of the zero complex $\{v_0\}$ is just v_0, see Remark 5. Therefore the point $(1,0)$ belongs to a unique interior. Furthermore, since the σ_1 was injective (from part 1) then every point of X belongs to a unique interior.
(4) You can check this pictorially.

Example 23. The goal of this example to describe a Δ-complex structure on $\mathbb{R}P^2$, the real projective plane of dimension 2.

$$\mathbb{R}P^2 \cong D^2 \Big/ \sim \text{ where } x \sim -x, \qquad x \in \partial D^2$$

This descriptions is taking the unit sphere in $\mathbb{R}^3$ and moding by the equivalence relation where we have identified antipidal points with each other. The space $\mathbb{R}P^2$ can also be identified as follows: each point of $\mathbb{R}P^2$ corresponds to a line through the origin in $\mathbb{R}^3$, I will discuss this more in class. The Δ-complex structure on $\mathbb{R}P^2$ is given in Figure 11.

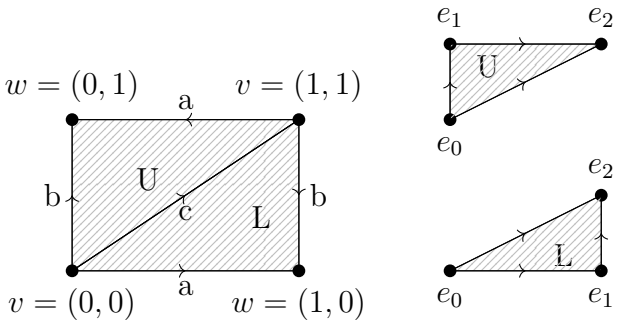


FIGURE 11. Δ-complex structure on $\mathbb{R}P^2$

The reason the vertices v and w get their coordinates is because this picture can be thought of as $[0,1] \times [0,1]$. The left hand side figure should look like a square (despite it not looking like one.) Table 1 gives the gluing data.

Dimension 0	Dimension 1	Dimension 2
$V : \{e_0\} \to \mathbb{R}P^2$ $V(e_0) = v$ $W : \{e_0\} \to \mathbb{R}P^2$ $W(e_0) = w$	$a : [f_0, f_1] \to \mathbb{R}P^2$ $a(t) = t \cdot \underbrace{(1,0)}_{w}$ $b : [f_0, f_1] \to \mathbb{R}P^2$ $b(t) = (1-t)\underbrace{(0,1)}_{w}$ $c : [f_0, f_1] \to \mathbb{R}P^2$ $c(f_0) = (1,1),\ c(f_1) = (0,0)$	$u : [e_0, e_1, e_2] \to \mathbb{R}P^2$ $u(e_0) = (0,0)$ $u(e_1) = (1,1)$ $u(e_2) = (0,1)$ $L : [e_0, e_1, e_2] \to \mathbb{R}P^2$ $L(e_0) = (0,0)$ $L(e_1) = (1,1)$ $L(e_2) = (1,0)$

TABLE 1. Δ-complex structure on $\mathbb{R}P^2$-Gluing Data.

Note that

$$\begin{aligned} u|_{[e_0,e_1]} &= L|_{[e_0,e_1]} = c \\ L|_{[e_1,e_2]} &= u|_{[e_0,e_2]} = b \\ L|_{[e_0,e_2]} &= u|_{[e_1,e_2]} \circ A = a \text{ where } A : [e_0, e_2] \to [e_1, e_2] \end{aligned}$$

5.1. **Δ-Chain Complex.** Given $\{\sigma_\alpha\}$- a Δ-complex structure on X we define the Δ-chain complex by:

$$C_k^\Delta := \mathbb{Z}\langle \sigma_\alpha : \Delta_\alpha \to X \mid \dim(\Delta_\alpha) = k \rangle = \left(\bigoplus_\alpha \mathbb{Z} \right)$$

where the boundary homomorphism is given by:

$$\delta_k^\Delta : C_k^\Delta \to C_{k-1}^\Delta$$
$$\delta_k^\Delta(\sigma_\alpha) : \sum_{i=0}^{k}(-1)^i \sigma_\alpha|_{[v_0,\ldots,\hat{v_i},\ldots,v_k]}.$$

Lemma 4. $\delta_k^\Delta \circ \delta_{k+1}^\Delta = 0$.

Lemma 4 implies that $(C_k^\Delta(X), \delta_k^\Delta)$ is indeed a chain complex called. The homology groups of this complex are called the *the Δ-homology groups of X* and are denoted by $H_k^\Delta(X) := \dfrac{\ker(\delta_k^\Delta)}{\mathrm{Im}(\delta_{k+1}^\Delta)}$. You will compute the Δ-homology groups of $\mathbb{R}P^2$ from Example 23 in your homework.

Example 24. [Example 23 continued] The groups of the Δ-complex are given by

$$C_0^\Delta(\mathbb{R}P^2) = \mathbb{Z}\langle v, w\rangle \cong \mathbb{Z}^2$$
$$C_1^\Delta(\mathbb{R}P^2) = \mathbb{Z}\langle a, b, c\rangle \cong \mathbb{Z}^3$$
$$C_2^\Delta(\mathbb{R}P^2) = \mathbb{Z}\langle u, L\rangle$$
$$0 \xrightarrow{\delta_3^\Delta} \mathbb{Z}\langle u, L\rangle \xrightarrow{\delta_2^\Delta} \mathbb{Z}\langle a, b, c\rangle \xrightarrow{\delta_1^\Delta} \mathbb{Z}\langle v, w\rangle \xrightarrow{\delta_0^\Delta} 0.$$

Next we are going to compute the kernel, image and the matrix of each δ_i^Δ using the data in Table 1.
-For $k = 0$ it is clear that $\mathrm{Im}(\delta_0^\Delta) = 0$ and $\ker(\delta_0^\Delta) = \mathbb{Z}\langle v, w\rangle$
-For $k = 1$:

$$\delta_1^\Delta(a) = a|_{[\hat{f_0}, f_1]} - a|_{[f_0, \hat{f_1}]} = w - v$$
$$\delta_1^\Delta(b) = b|_{[\hat{f_0}, f_1]} - b|_{[f_0, \hat{f_1}]} = w - v$$
$$\delta_1^\Delta(a) = a|_{[\hat{f_0}, f_1]} - a|_{[f_0, \hat{f_1}]} = v - v = 0$$
$$(\delta_1^\Delta)_{2\times 3} = \begin{array}{c} \\ v \\ w \end{array}\!\!\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathrm{Im}(\delta_1^\Delta) = \mathbb{Z} \underset{\text{reduce}}{\longrightarrow} H_0^\Delta(\mathbb{R}P^2) \cong \frac{\mathbb{Z}\oplus\mathbb{Z}}{\mathbb{Z}\oplus 0} \cong \mathbb{Z}$$

In fact one can show $\ker(\delta_1^\Delta) = \mathbb{Z}\langle c, a + b\rangle$. Regardless of this, the matrix of δ_1^Δ shows $\ker(\delta_1^\Delta) \cong \mathbb{Z}\oplus\mathbb{Z} = \mathbb{Z}^2$.

-For $k = 2$ one should certainly consult Figure 11 as the maps are much more clear pictorially.

$$\delta_2^\Delta : \mathbb{Z}\langle u, L\rangle \to \mathbb{Z}\langle a, b, c\rangle$$
$$\delta_2^\Delta(u) = u|_{[\hat{e_0}, e_1, e_2]} - u|_{[e_0, \hat{e_1}, e_2]} + u|_{[e_0, e_1, \hat{e_2}]} = a - b + c$$
$$\delta_2^\Delta(L) = L|_{[\hat{e_0}, e_1, e_2]} - L|_{[e_0, \hat{e_1}, e_2]} + L|_{[e_0, e_1, \hat{e_2}]} = b - a + c$$

$$(\delta_2^\Delta)_{3\times 2} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{cc} u & v \end{array} \\ \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} \end{array} \to \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \underset{*}{\Rightarrow} \mathrm{Im}(\delta_2^\Delta) \cong \mathbb{Z} \oplus 2\mathbb{Z}$$

$$H_1^\Delta(\mathbb{R}P^2) \cong \frac{\ker(\delta_1^\Delta)}{\mathrm{Im}(\delta_2^\Delta)} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\mathbb{Z} \oplus 2\mathbb{Z}} \cong \mathbb{Z}_2$$

In particular we have shown that $\ker(\delta_2^\Delta) = 0$ therefore

$$H_2^\Delta(\mathbb{R}P^2) \cong \frac{0}{0} \cong 0$$

Note: I believe the implication $*$ was discussed in class. You do not need any facts for this, from the entries of the matrix it is clear the image is isomorphic to $\mathbb{Z} \oplus 2\mathbb{Z}$.

To summerize this example, we have computed each homology group i.e. its decomposition as a finitely generated ableian group and therefore we have computed the Betti numbers of $\mathbb{R}P^2$ (with a Δ-complex structure.).

$$H_k^\Delta(\mathbb{R}P^2) = \begin{cases} \mathbb{Z}\ ; & k = 0 \\ \mathbb{Z}_2\ ; & k = 1 \\ 0\ ; & \text{otherwise} \end{cases}$$
$$\beta_0(\mathbb{R}P^2) = 1$$
$$\beta_k(\mathbb{R}P^2) = 0 \text{ for } k \neq 0$$

Remark 6. As Corollary 3 indicates, the zeroth homology group keeps track of path connected components of X. A rough and somewhat imprecise intuition for the free parts of the first and second homology groups are as follows: the free part of the first homology group keeps track of "2-dimensional holes" and the free part of the second homology group keeps track of "3-dimensional" holes. Consider the torus T. We know $H_1(T) \cong \mathbb{Z}^2$. You can see this pictorially below. Moreover, $H_2(T) \cong \mathbb{Z}$ and that's because if you were to fill the torus with jelly, or your filling of choice!, there is only one such hole. On the other hand, the filled torus T' will have $H_1(T') \cong \mathbb{Z}$ and $H_2(T) \cong 0$. As another example of a "3-dimensional hole", if X denotes two spheres joined at a single point, then $H_2(X) \cong \mathbb{Z}^2$.

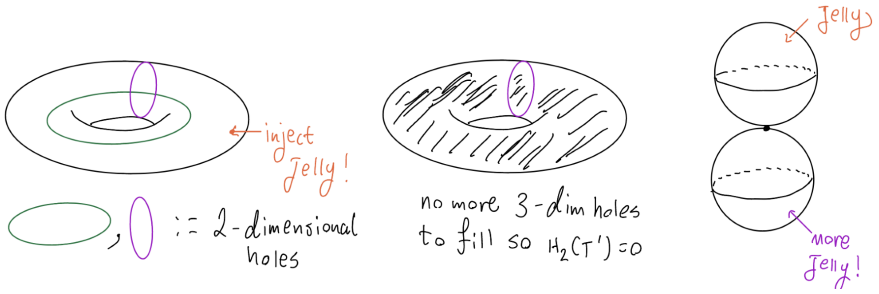


FIGURE 12. Rough Intuition for first and second homology groups

5.2. **Euler Characteristic.** Start with C_*, a chain complex of finitely generated $\mathbb{Z}$-modules. The Euler characteristic of C_* is the alternating sum of the ranks of the modules appearing in C_*.

Definition 19 (Euler Characteristic)**.** Given a chain complex $(C_p, \delta_p)_{p\in\mathbb{Z}}$ of finitely generated $\mathbb{Z}$-modules i.e. $C_k \cong \mathbb{Z}^{c_k}$ for $c_k \geq 0$ we define

$$\chi(C_*) = \sum_{k=0}^{\infty} (-1)^k c_k. \tag{5.1}$$

The quantity $\chi(C_*)$ is called *the Euler characteristic of* C_*.

In Definition 19 if we specifically looks at a simplicial complex $\mathcal{K}$ and its chain complex $C_*(\mathcal{K})$ Equation (5.1) becomes

$$\chi(C_*(\mathcal{K})) = \chi(\mathcal{K}) = \sum_{k=0}^{\infty} (-1)^k \operatorname{rank}(C_k(\mathcal{K}, \mathbb{Z})),$$

where $\operatorname{rank}(C_k(\mathcal{K}, \mathbb{Z}))$ corresponds to the rank of the degree k part of C_* as a $\mathbb{Z}$-module. Similarly one can define the Euler characteristic via a Δ-complex structure on X.

Note that a triangulation Definition 17 is an example of a Δ-complex structure.

Example 25. Suppose $\mathcal{K} \cong S^2$, that is $\mathcal{K}$ is any triangulation of S^2. Then $\chi(S^2)$ is computed as $V - E + F$; Vertices-Edges+Faces.

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $\chi = V - E + F$
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

FIGURE 13. Euler Characteristic of S^2 does not depend on triangulation.

Friday February 6th:

"Getting ahead in a difficult profession requires avid faith in yourself. You must be able to sustain yourself against staggering blows... That is why some people with mediocre talent, but with great inner drive, go so much further than people with vastly superior talent."
-Sophia Loren.

Theorem 5.1. *Given a simplicial complex $\mathcal{K}$ and its chain complex $C_*(\mathcal{K})$ we have*

$$\chi(C_*(\mathcal{K}))) = \sum_{k=0}^{\infty}(-1)^k \operatorname{rank}(C_k(\mathcal{K})) = \sum_{k=0}^{\infty}(-1)^k \beta_k(\mathcal{K})$$

Proof. The proof follows from a repeated use of Rank-Nullity theorem and keeping track of the sum. Note that

$$\operatorname{rank}(C_k(\mathcal{K}))) = \operatorname{rank}(\ker(\delta_k)) + \operatorname{rank}(\operatorname{Im}(\delta_k)).$$

□

As an application/corollary of Theorem 5.1 we get that if one can show the betti numbers $\beta_k(\mathcal{K})$ are invariant under homotopy equivalence then we get that $\chi(\mathcal{K})$ is a topological invariant.

6. Homology, a topological invariant

The main goal of this section is to show either types of homology groups, the ones coming from a Δ-complex structure denoted $H_*^{\Delta}(X)$ or the ones coming from a simplicial complex denoted $H_*^{\text{simp}}(X)$, are topological invariants. Similar to the fundamental group of X, where $\pi_1(X, x_0)$ is an invariant of X under homotopy equivalence, we would like to know if the same is true for Homology groups.

To make this precise, we need to understand: given a map between topological spaces $f : X \to Y$ can we get an induced homomorphism between homology groups i.e. $f_* :$

$H_*(X) \to H_*(Y)$? The answer is of course yes and we shall make this precise during the rest of this section.

Definition 20. Let $A_* = (A_k, a_k)_{k\in\mathbb{Z}}$ and $B_* = (B_k, \beta_k)_{k\in\mathbb{Z}}$ be two chain complexes. A family of homomorphisms $f_* = \{f_k : A_k \to B_k\}$ is called a chain complex homomorphism of A_* and B_* if and only if the squares of the following diagram commute for all k.

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{a_{k+2}} & A_{k+1} & \xrightarrow{a_{k+1}} & A_k & \xrightarrow{a_k} & A_{k-1} & \xrightarrow{a_{k-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{k+1}} & \circlearrowright & \downarrow{\scriptstyle f_k} & \circlearrowright & \downarrow{\scriptstyle f_{k-1}} & & \\
\cdots & \xrightarrow{\beta_{k+2}} & B_{k+1} & \xrightarrow{\beta_{k+1}} & B_k & \xrightarrow{\beta_k} & B_{k-1} & \xrightarrow{\beta_{k-1}} & \cdots
\end{array}$$

Lemma 5. *A chain complex homomormphism $f_* : (A_*, a_*) \to (B_*, \beta_*)$ induces a group homomorphism $\hat{f}_* : H_*(A_*) \to H_*(B_*)$. More precisely for each $k \in \mathbb{Z}$, there exists a group homomorphism, induced by f_* where*

$$\begin{aligned} \hat{f}_k &: H_k(A_K) \to H_k(B_k), \\ \hat{f}_k &: \frac{\ker(a_k)}{\mathrm{Im}(a_{k+1})} \to \frac{\ker(b_k)}{\mathrm{Im}(b_{k+1})}. \end{aligned} \tag{6.1}$$

Proof. One should focus on the middle part of the diagram in Definition 20 i.e. this proof involves the map f_k and its source and target. Using the following two claims we prove the result and then we prove the claims. For any k we have

(1) $f_k(\ker(a_k)) \subset \ker(b_k)$
(2) $f_k(\mathrm{Im}(a_{k+1})) \subset \mathrm{Im}(b_{k+1})$

Firstly, (1) implies that $f_k|_{\ker(a_k)} : \ker(a_k) \to \ker(b_k)$ is a map with $\ker(a_k)$ as its domain and $\ker(b_k)$ is its target. Further note that $\mathrm{Im}(a_{k+1}) \subseteq \ker(a_k)$ so it makes sense to apply $f_k|_{\ker(a_k)}$ to $\mathrm{Im}(a_{k+1})$. Item (2) implies there exists a well-defined quotient homomorphism

$$\hat{f}_k : \frac{\ker(a_k)}{\mathrm{Im}(a_{k+1})} \to \frac{\ker(b_k)}{\mathrm{Im}(b_{k+1})}.$$

To be precise, there are two interpretation for the well-definedness of a map: is the image actually in the target we claim it lands in? In this case the answer is yes by (1). Do different representatives of the same equivalence class go to the same image in the target? The answer is yes by (2) and I will leave this as a homework problem. To prove items (1) and (2) we do what is called "diagram chasing".

Proof of (1): Let $t \in \ker(a_k)$, that is $a_k(t) = 0$. We wish to show $f_k(t) \in \ker(b_k)$, that is $b_k(f_k(t)) = 0$. However, the commutativity of the second square in Definition 20 yields $b_k(f_k(t)) = f_{k-1}(a_k(t)) = f_{k-1}(0) = 0$ and 1 follows.

I leave the second item to you but the proof should be identical to (1). □

7. Singular Homology

Let X be any topological space. A set of singular simplicies of dimension k in X is a family of all continuous maps:

$$\sigma_k = \left\{ \sigma : \underbrace{\Delta^k}_{[v_0,\dots,v_k]} \to X \right\}$$

where Δ^k's are considered modulo linear homeomorphisms preserving orientation.

Definition 21 (Singular Homology of X)**.** Singular chain complex of X is defined via the following data: the groups are the free abelian groups generated by σ's, that is formal sums $\sum_{i=1}^{t} n_i\sigma_i$ where $n_i \in \mathbb{Z}$ and the maps are given by an alternating sum of restriction to faces. More precisely:

$$C_k^{\text{sing}}(X,\mathbb{Z}) = C_k^{\text{sing}}(X) = \mathbb{Z}\langle\sigma_k\rangle = \bigoplus_{\sigma\in\sigma_k}\mathbb{Z}$$

$$\delta_k^{\text{sing}} : C_k^{\text{sing}}(X;,\mathbb{Z} \longrightarrow C_{k-1}^{\text{sing}}(X;\mathbb{Z})$$

$$\delta_k^{\text{sing}}(\tau) = \sum_{i=0}^{k}(-1)^i\tau|_{[v_0,v_1,\dots,\hat{v_i},\dots,v_k]}$$

One can show, similar to Δ-complexes that $\delta_k^{\text{sing}}\circ\delta_{k+1}^{\text{sing}} = 0$. To nobody's surprise, the singular homology is then defined as

$$H_k^{\text{sing}}(X;\mathbb{Z}) = H_k^{\text{sing}}(X) := \frac{Z_k^{\text{sing}}(X)}{B_k^{\text{sing}}(X)}$$

where $Z_k^{\text{sing}}(X) = \ker(\delta_k^{\text{sing}})$ and $B_k^{\text{sing}} = \text{Im}(\delta_{k+1}^{\text{sing}})$.

There is a very nice discussion in [Mun84, p.162-163] about singular homology that I highly recommend you read. It also explain exactly why δ_k^{sing} actually lands in C_{k-1}^{sing} i.e. why is it that $\tau|_{[v_0,v_1,\dots,\hat{v_i},\dots,v_k]}$ an honest element of C_{k-1}^{sing}. (It is not obvious from Definition 21.)

Monday February 9th:

"I have to constantly re-identify myself to myself, reactivate my own standards, my own convictions about what I'm doing and why."

-Nina Simone

Example 26. [Singular Homology of a point] Let $X = \{\text{point}\}$. Given any k-simplex Δ^k there is exactly one map, namely the trivial map from Δ^k to X. Therefore up to linear homeomorphisms preserving orientation there is exactly one map $\sigma_k : \Delta^k \to X$. This means any possible map you can write is the trivial map. Thus for every $k \geq 0$ we have $C_k^{\text{sing}}(X) = \mathbb{Z}\langle\sigma_k\rangle \cong \mathbb{Z}$. So far we have

$$\cdots \longrightarrow \underbrace{\mathbb{Z}}_{C_k} \xrightarrow{\delta_k^{\text{sing}}} \underbrace{\mathbb{Z}}_{C_{k-1}} \xrightarrow{\delta_{k-1}^{\text{sing}}} \cdots \longrightarrow \underbrace{\mathbb{Z}}_{\mathbb{Z}\langle\sigma_2\rangle} \xrightarrow{\delta_2^{\text{sing}}} \underbrace{\mathbb{Z}}_{\mathbb{Z}\langle\sigma_1\rangle} \xrightarrow{\delta_1^{\text{sing}}} \underbrace{\mathbb{Z}}_{C_0} \xrightarrow{\delta_0^{\text{sing}}=0} 0$$

$$\delta_1^{\text{sing}}(\sigma_1) = \sigma_1|_{[v_1]} - \sigma_1|_{[v_0]} = 0$$

because the trivial map has the same value when restricted to any face.

$$\delta_1^{\text{sing}}(\sigma_2) = \sigma_2|_{[v_1,v_2]} - \sigma_1|_{[v_0,v_2]} + \sigma|_{[v_0,v_1]} = \sigma_2|_{[v_1,v_2]} = \sigma_1 \Rightarrow \sigma_2^{\text{sing}} = \text{Id}.$$

One can generalize this to any even or odd index to get

$$\delta_k^{\text{sing}} = \begin{cases} 0 & ; k - \text{odd} \\ \text{Id} & ; k - \text{even} \end{cases}$$

$$\cdots \longrightarrow \underbrace{\mathbb{Z}}_{C_4} \xrightarrow{\text{Id}} \underbrace{\mathbb{Z}}_{C_3} \xrightarrow{0} \underbrace{\mathbb{Z}}_{\mathbb{Z}\langle\sigma_2\rangle} \xrightarrow{\text{Id}} \underbrace{\mathbb{Z}}_{\mathbb{Z}\langle\sigma_1\rangle} \xrightarrow{0} \underbrace{\mathbb{Z}}_{C_0} \xrightarrow{0} 0$$

$$H_k^{\text{sing}}(\{\text{pt}\} \cong \begin{cases} \mathbb{Z} & ;\ k = 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Lemma 6. *If X is non-empty and path-connected then $H_0^{sing}(X) \cong \mathbb{Z}$.*

Proof. Recall $C_0(X) = \langle \tau_\alpha \rangle$ where $\tau_\alpha : \Delta_0 \to X$, where $\Delta_0 = \{v_0\}$. So any element of $C_0(X)$ is a formal $\mathbb{Z}$-linear combination of such τ_α. Define the map

$$\epsilon : C_0(X) \to \mathbb{Z},$$
$$\epsilon(\sum_\alpha k_\alpha \tau_\alpha) = \sum_\alpha k_\alpha.$$

Notice that ϵ is an onto map and $\mathbb{Z}$-linear. Since X is not empty, otherwise this whole discussion would be empty, X has at least one point, thus C_0 has at least one generator and therefore ϵ is onto. That is to say for any $x_0 \in X$, there exists τ_0 mapping v_0 to x_0. Therefore for any $k \in \mathbb{Z}$, $\epsilon(k \cdot \tau_0) = k$. (In fact, slightly more is true: ϵ is an epimorphism, if you know what that is from category theory, a consequence of which is that ϵ is onto.)

It suffices to show $\ker(\epsilon) = \text{Im}(\delta_1)$. If true, then the first isomorphism theorem implies

$$\frac{C_0(X)}{\ker(\epsilon)} \cong \mathbb{Z}$$
$$\frac{C_0(X)}{\ker(\epsilon)} = \frac{C_0(X)}{\text{Im}(\delta_1)} = \frac{\delta_0}{\text{Im}(\delta_1)} = H_0^{\text{sing}}(X).$$

To this end, let $\gamma \in \text{Im}(\delta_1)$ i.e. γ is the δ_1 applies to some element of $C_1(X)$, which recall are formal linear combinations of singular 1-simplicies thus

$$\gamma = \delta_1 \left(\sum_\alpha k_\alpha \sigma_\alpha \right) \text{ where } \sigma_\alpha : [v_0, v_1] \to X \text{ for all } \alpha.$$

Using the definitions of δ_1 and ϵ we get

$$\epsilon(\gamma) = \epsilon\left(\delta_1\left(\sum_\alpha k_\alpha \sigma_\alpha\right)\right) = \epsilon\left(\sum_\alpha k_\alpha(\delta_1(\sigma_\alpha))\right) = \epsilon\left(\sum_\alpha k_\alpha(\sigma_\alpha|_{\{v_1\}} - \sigma_\alpha|_{\{v_0\}})\right)$$
$$= \left(\sum_\alpha k_\alpha(\underbrace{\epsilon(\sigma_\alpha|_{\{v_1\}})}_{1} - \underbrace{\epsilon(\sigma_\alpha|_{\{v_0\}})}_{1})\right) = 0 \Rightarrow \text{Im}(\delta_1) \subseteq \ker(\epsilon).$$

Let $\tau = \sum_\alpha k_\alpha \tau_\alpha \in \ker(\epsilon) \subseteq C_0(X)$ which implies $\sum_\alpha k_\alpha = 0$, therefore k_α's are fixed going forward, and $\tau_\alpha : \{v_0\} \to X$ with its image $\tau_\alpha(v_0) = x_\alpha$, which is a point of X. Next, we will define an element, γ of $C_1(X)$ whose image under δ_1 equals τ which finishes the proof. Fix a point $v \in X$. Since X is path-connected, for any point of X, in particular for the points x_α, there exists a path between v and x_α. Thus for each index α

$$\gamma_\alpha : [0,1] \to X \quad \text{where } \gamma_\alpha(0) = v, \;\; \gamma_\alpha(1) = x_\alpha.$$

Notice that $\gamma = \sum_\alpha k_\alpha \gamma_\alpha$ is an element of $C_1(X)$ and is indeed our desired element because

$$\begin{aligned}\delta_1(\gamma) &= \delta_1\left(\sum_\alpha k_\alpha \gamma_\alpha\right) = \sum_\alpha k_\alpha \delta_1(\gamma_\alpha) = \sum_\alpha k_\alpha(\gamma_\alpha|_{\{1\}} - \gamma_\alpha|_{\{0\}}) \\ &= \sum_\alpha k_\alpha \underbrace{(\gamma_\alpha|_{\{1\}})}_{\tau_\alpha} - (\sum_\alpha k_\alpha \underbrace{(\gamma_\alpha|_{\{0\}})}_{\tau_v} = \sum_\alpha k_\alpha \tau_\alpha - \underbrace{(\sum_\alpha k_\alpha)}_{0}\tau_v = \tau.\end{aligned}$$

Notice that the last line uses two identifications of maps as follows: for each α the map $\gamma_\alpha|_{\{1\}}$ is, up to a linear homeomorphism, the same map as $\tau_\alpha : \{v_0\} \to X$. The image of $\gamma_\alpha|_{\{1\}}$ and τ_α are exactly the same, it is x_α and the domains can be identified, via a linear homeomorphism, as the same. Since singular simplicies are defined up to linear homeomorphisms preserving orientation, these maps are the same for us. An identical reasoning explains why $\gamma_\alpha|_{\{0\}}$ and $\tau_v : \{v_0\} \to X$, where $\tau_v(\{v_0\}) = v$, are the same map. (In class, I was identifying $\gamma_\alpha|_{\{1\}}$ and τ_α as "x_α" and just writing that, however we can't really add points of X since it does not have a group structure. So x_α was being used as short-hand for both of the maps, or if you like as a "representative for the equivalence class of both of the maps".) □

Wednesday February 11th:

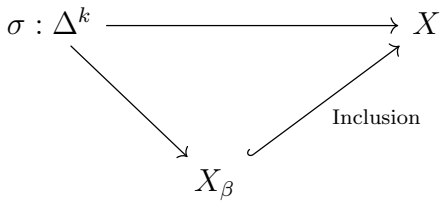
"Your mind will answer most questions if you learn to relax."

-William S. Burroughs.

Theorem 7.1. *Let $\{X_\beta\}$ be a collection of path-connected components of X. For every $k \geq 0$ we have*

$$H_k(X) \cong \bigoplus_\beta H_k(X_\beta). \tag{7.1}$$

Proof. Fix k and let $\sigma : \Delta^k \to X$ be a k-simplex. Notice the image of this k-simplex, $\sigma(\Delta^k)$, belongs to a unique path connected component X_β. Therefore for each k-simplex there exists a unique β such that:



Therefore any element $\tau \in C_k^{\text{sing}}(X)$ has a unique decomposition indexed by the path-connected components $\tau = \sum_\beta \tau_\beta$ therefore

$$C_k^{\text{sing}}(X) = \bigoplus_\beta C_k^{\text{sing}}(X_\beta).$$

This observation, that the singular (chain) complex structure of X induces a singular (chain) complex structure on path-connected components X_β of X is the crux of the proof because

$$\delta_k(\tau) = \sum_\beta \delta_k^\beta \tau_\beta \Rightarrow \ker(\delta_k) = \bigoplus_\beta \ker(\delta_k^\beta) \text{ and } \operatorname{Im}(\delta_{k+1}) = \bigoplus_\beta \operatorname{Im}(\delta_{k+1}^\beta).$$

Finally, observe that

$$H_k^{\text{sing}}(X) = \frac{\ker(\delta_k)}{\operatorname{Im}(\delta_{k+1})} = \frac{\bigoplus_\beta \ker(\delta_k^\beta)}{\bigoplus_\beta \operatorname{Im}(\delta_{k+1}^\beta)} \cong \bigoplus_\beta \left(\frac{\ker(\delta_k^\beta)}{\operatorname{Im}(\delta_{k+1}^\beta)} \right) = \bigoplus_\beta H_k^{\text{sing}}(X_\beta). \qquad \square$$

Corollary 3. *For a topological space X where $\{X_\beta\}$ are the path connected components of X,*

$$H_0^{sing}(X) \cong \bigoplus_{X_\beta} \mathbb{Z}$$

$$\beta_0(X) = \text{ number of path connected components of } X.$$

To put Corollary 3 to words, $H_0(X)$ is the free abelian group generated by the set of path-connected components of X. Moreover the path connected components of X give a grading of $H_0^{\text{sing}}(X)$.

7.1. **Reduced Singular Homology.** It is mildly disturbing that the zeroth singular homology group of a point is $\mathbb{Z}$. For this, and a multitude of other reasons, it is desirable that this homology group be 0. To accomplish this we will define another notion of homology which is designed for this purpose.

Definition 22 (Reduced Singular Homology)**.** The reduced singular homology group of a topological space X, denoted by $\tilde{H}_k^{\text{sing}}$, is the homology of the (reduced) chain complex:

$$\cdots \longrightarrow C_k^{\text{sing}} \xrightarrow{\delta_k^{\text{sing}}} C_{k-1}^{\text{sing}} \xrightarrow{\delta_{k-1}^{\text{sing}}} \cdots \longrightarrow C_2^{\text{sing}} \xrightarrow{\delta_2^{\text{sing}}} C_1^{\text{sing}} \xrightarrow{\delta_1^{\text{sing}}} C_0^{\text{sing}} \xrightarrow{\epsilon = \tilde{\delta_0}} \underbrace{\mathbb{Z}}_{C_{-1}^{\text{sing}}(X)} \longrightarrow 0$$

where the map ϵ is defined as $\epsilon(\sigma_0) \to 1$. From the definition of epsilon, it follows that $\operatorname{Im}(\delta_1) \subseteq \ker(\epsilon)$.

Lemma 7 (Relationship between $H_*^{\text{sing}}(X)$ and $\tilde{H}_*^{\text{sing}}$)**.**

$$H_k^{sing}(X) = \begin{cases} \tilde{H}_k^{sing} \ ; \ k > 0. \\ \tilde{H}_0^{sing} \oplus \mathbb{Z} \ ; \ k = 0. \end{cases}$$

As an immediate consequence of Lemma 7 we get:

Corollary 4. *If X is path connected, then $\tilde{H}_0(X) \cong 0$.*

8. Homomorphisms Induced by Homotopic maps

Definition 23 (Homotopic maps)**.** For topological spaces X and Y, the maps $f, g : X \to Y$ are (freely) homotopic, denoted $f \simeq g$ if and only if there exists $F : X \times I \to Y$ such that $F|_{X\times\{0\}} = f$ and $F|_{X\times\{1\}} = g$. Note that $I = [0, 1]$ denotes the unit interval and a different notation would be to say $F(t, 0) = f$ and $F(t, 1) = g$.

Definition 24 (Homotopy Equivalent Spaces)**.** Two spaces X and Y are homotopy equivalent, denoted $X \simeq Y$, if and only if there exists maps $f : X \to Y$ and $g : Y \to X$ satisfying

$$g \circ f \simeq \mathrm{id}_X, \quad f \circ g \simeq \mathrm{id}_Y.$$

The maps f and g are called homotopy equivalences.

Homotopy equivalence is a weaker notion than Homeomorphism. It is straightforward to show the if X and Y are homeomorphic, i.e. $X \cong Y$ then $X \simeq Y$. However, the backwards direction is not true. If $X \simeq Y$ does not imply $X \cong Y$.

Example 27. Suppose $X = \mathbb{R}^m$ and $Y = \{\mathrm{pt}\}$. Then $\mathbb{R}^m \simeq \{\mathrm{pt}\}$. One can show the maps f and g

$$f : \{\mathrm{pt}\} \hookrightarrow \{0\} \subseteq \mathbb{R}^m g : \mathbb{R}^m \to \{\mathrm{pt}\}$$

are homotopy equivalences.

Our goal for the rest of this section is to show that homotopic spaces have the same (reduced) singular homology groups i.e.

$$X \simeq Y \Rightarrow H_k^{\mathrm{sing}}(X) \cong H_k^{\mathrm{sing}}(Y) \text{ for all } k \geq 0.$$

Recall that in Lemma 5 we showed a chain complex homomorphism leads to a homomorphism between the homology groups. Next, we want to show a map between two topological spaces induced a map on their chain complex groups. That is for all $k \geq 0$ we have

$$f : X \to Y \underset{*}{\text{ induces }} f_k : C_k(X) \to c_k(Y) \text{ Lemma 5 induces } f_k : H_k(X) \to H_k(Y). \tag{8.1}$$

The $*$ implication is what we are proving next.

Wednesday February 18th:

"In order to create, the need to express has to be bigger than the fear."

-Juliette Binoche.

Lemma 8. *Given a map $f : X \to Y$ we can define an induced homomorphism $f_* : C_*(X) \longrightarrow C_*(Y)$ on the groups of their chain complexes given by*

$$f_k : C_k(X) \longrightarrow C_k(Y)$$
$$f_k(\sigma) := f \circ \sigma$$
$$\Delta^k \xrightarrow{\sigma} X \xrightarrow{f} Y.$$

Proof. To show the map f_* is indeed a map of chain complexes we need to show the commutativity of the following diagram for all k:

$$\begin{array}{ccccccc}
\longrightarrow & C_k(X) & \xrightarrow{\delta_k^X} & C_{k-1}(X) & \longrightarrow \\
& \downarrow{\scriptstyle f_k} & & \downarrow{\scriptstyle f_{k-1}} & \\
\longrightarrow & C_k(Y) & \xrightarrow[\delta_k^Y]{} & C_{k-1}(Y) & \longrightarrow
\end{array}$$

For any $\tau \in C_k(X)$ where $\tau = \sum_\alpha k_\alpha \sigma_\alpha$ we have

$$\delta_k^Y \circ f_k(\tau) = \delta_k^Y(f_k(\sum_\alpha k_\alpha \sigma_\alpha)) = \delta_k^Y(\sum_\alpha k_\alpha(f_k \circ \sigma_\alpha)) = \sum_\alpha k_\alpha(\delta_k^Y(f_k \circ \sigma_\alpha))$$

$$\sum_\alpha k_\alpha(\sum_{i=0}^{k}(-1)^k f_k \circ \sigma_\alpha|_{[e_0,\dots,\hat{e_i},\dots,e_k]}) \underset{*}{=} \sum_\alpha k_\alpha f_{k-1}(\delta_k^X \sigma_\alpha)$$
$$= f_{k-1}(\delta_k^X \tau) = (f_{k-1} \circ \delta_k^X)(\tau).$$

where the equality $*$ follows from the definition of f_{k-1}. □

By Lemma 8 and Equation (8.1), any map $f : X \to Y$ induces a homomorphism on homology.

Lemma 9. *Given $f : X \to Y$ and $g : Y \to \mathbb{Z}$ we have that $(g \circ f)_* = g_* \circ f_*$. Diagrammatically this yields:*

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
& \searrow & & \nearrow & \\
& & g\circ f & &
\end{array}
\qquad
\begin{array}{ccccc}
H_*(X) & \xrightarrow{f_*} & H_*(Y) & \xrightarrow{g_*} & H_*(Z) \\
& \searrow & & \nearrow & \\
& & (g\circ f)_* & &
\end{array}$$

Proof. Homework. □

Theorem 8.1. *If $f, g : X \to Y$ are homotopics, that is $f \simeq g$, then for all k we have $f_k = g_k$ as maps where*

$$f_k : C_k(X) \longrightarrow C_k(Y)$$
$$g_k : C_k(X) \longrightarrow C_k(Y)$$

Before proving the theorem we will use it to prove the following corollary.

Corollary 5. *If $X \simeq Y$ then $H_k(X) \cong H_k(Y)$ for all k.*

Proof. By Definition 24 there exists $F : X \longrightarrow Y$ and $G : Y \longrightarrow X$ such that $F \circ G \simeq \mathrm{id}_Y$ and $G \circ F \simeq \mathrm{id}_X$. For all k Theorem 8.1 implies their induced maps on the Homology groups are the same.

$$(F \circ G)_k = (\mathrm{id}_Y)_k : H_k(Y) \longrightarrow H_k(Y) \quad (G \circ F)_k = (\mathrm{id}_X)_k : H_k(X) \longrightarrow H_k(X) \tag{8.2}$$

Notice that $(\mathrm{id}_Y)_k$ induces the identity map on Homology groups thus $(\mathrm{id}_Y)_k = \mathrm{id}_{H_k(Y)}$ and similarly $(\mathrm{id}_X)_k = \mathrm{id}_{H_k(X)}$. By Lemma 9 we get:

$$F_k \circ G_k \underset{9}{=} (F \circ G)_k \underset{8.2}{=} \mathrm{id}_{H_k(Y)} \text{ and } G_k \circ F_k \underset{9}{=} (G \circ F)_k \underset{8.2}{=} \mathrm{id}_{H_k(X)}$$

Therefore $(F_k)^{-1} = G_k$ which implies $H_k(X) \cong H_k(Y)$. □

Remark 7. A note on Functionality of Homology.

In order to prove Theorem 8.1 we need the concept of a chain homotopy.

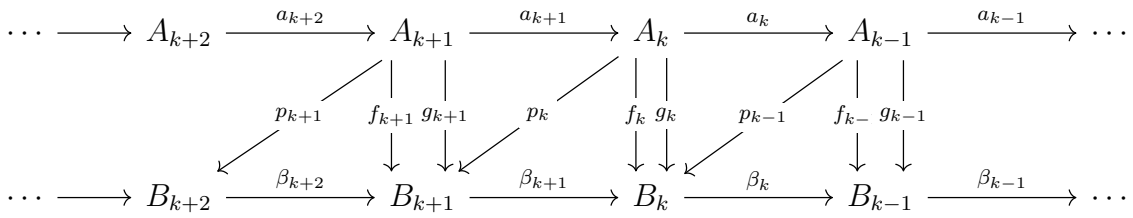
Definition 25 (Chain Homotopy)**.** Given chain complexes (A_*, a_*) and (B_*, β_*) and a pair of chain homomorphisms $f_*, g_* : (A_*, a_*) \longrightarrow (B_*, b_*)$, we say that f_*, g_* are chain homotopic if and only if there exists homomorphisms

$$p_k : A_k \longrightarrow B_{k+1}$$

satisfying the following relation:

$$f_k - g_k = b_{k+1} \circ p_k + p_{k-1} \circ a_k.$$

Diagrammatically this looks like



The collection of maps p_* is called a chain homotopy between f_* and g_*.

Note in Definition 25, the map p_k increases the homological degree by 1. Moreover the focus of the diagram are the parallelograms as opposed to square (the situation for a diagram to commute.)

Friday February 20th:

"Maybe that's enlightenment enough: to know that there is no final resting place of the mind; no moment of smug clarity. Perhaps wisdom...is realizing how small I am, and unwise, and how far I have yet to go."

-Anthony Bourdain.

Lemma 10. *If f_* are g_* are chain homotopic then they induce the same map on homology. Let the induced map on homology be denoted as $H_*(f_*)$ and $H_*(g_*)$ respectively, then*

$$H_*(f_*) : H_*(A_*) \longrightarrow H_*(B_*)$$
$$H_*(g_*) : H_*(A_*) \longrightarrow H_*(B_*)$$
$$H_*(g_*) = H_*(f_*)$$

We are now in position to prove Theorem 8.1.

Proof. Suppose $f, g : X \to Y$ are homotopic which recall means there exists

$$F : X \times [0,1] \longrightarrow Y \text{ such that } \begin{cases} F(x,0) = f(x) \\ F(x,1) = g(x). \end{cases} \tag{8.3}$$

By Lemma 8 implies f and g induce maps on chain complexes $C_*(X)$ and $C_*(Y)$ which yields the following diagram so far:

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_{k+2}(X) & \xrightarrow{\delta^X_{k+2}} & C_{k+1}(X) & \xrightarrow{\delta^X_{k+1}} & C_k(X) & \xrightarrow{\delta^X_k} & C_{k-1}(X) & \xrightarrow{\delta^X_{k-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{k+2}}\ \downarrow{\scriptstyle g_{k+2}} & & \downarrow{\scriptstyle f_{k+1}}\ \downarrow{\scriptstyle g_{k+1}} & & \downarrow{\scriptstyle f_k}\ \downarrow{\scriptstyle g_k} & & \downarrow{\scriptstyle f_{k-1}}\ \downarrow{\scriptstyle g_{k-1}} & & \\
\cdots & \longrightarrow & C_{k+2}(Y) & \xrightarrow{\delta^Y_{k+2}} & C_{k+1}(Y) & \xrightarrow{\delta^Y_{k+1}} & C_k(Y) & \xrightarrow{\delta^Y_k} & C_{k-1}(Y) & \xrightarrow{\delta^Y_{k-1}} & \cdots
\end{array}$$

Observe that by Lemma 10 it suffices to construct a chain homotopy between f_* and g_*. The conclusion of Lemma 10 then is our desired conclusion. This means we need to construct

$$p_k : C_k(X) \longrightarrow C_{k+1}(Y)$$

that satisfies Definition 25 and our desired diagram will look like

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & C_{k+2}(X) & \xrightarrow{\delta^X_{k+2}} & C_{k+1}(X) & \xrightarrow{\delta^X_{k+1}} & C_k(X) & \xrightarrow{\delta^X_k} & C_{k-1}(X) & \xrightarrow{\delta^X_{k-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{k+2}}\ \downarrow{\scriptstyle g_{k+2}} & \swarrow{\scriptstyle p_{k+1}} & \downarrow{\scriptstyle f_{k+1}}\ \downarrow{\scriptstyle g_{k+1}} & \swarrow{\scriptstyle p_k} & \downarrow{\scriptstyle f_k}\ \downarrow{\scriptstyle g_k} & \swarrow{\scriptstyle p_{k-1}} & \downarrow{\scriptstyle f_{k-1}}\ \downarrow{\scriptstyle g_{k-1}} & & \\
\cdots & \longrightarrow & C_{k+2}(Y) & \xrightarrow{\delta^Y_{k+2}} & C_{k+1}(Y) & \xrightarrow{\delta^Y_{k+1}} & C_k(Y) & \xrightarrow{\delta^Y_k} & C_{k-1}(Y) & \xrightarrow{\delta^Y_{k-1}} & \cdots
\end{array}$$

where p_i's are dotted because we haven't constructed them yet. The map p_* has a cool name, it is called the prism operator and it is constructed using the the F in Equation (8.3). The basic idea of p_k is as follows: given a k-simplex of X i.e.a generator of $C_k(X)$, say $\tau : \Delta^k \longrightarrow X$, we need to produce $k+1$-simplicies of Y in such a way that Definition 25 is satisfied. Given a k-simplex τ of X, we will perceive it inside the space $X \times [0,1]$ by viewing it as $\tau \times \mathrm{id}$ so its "shape remains the same inside the cylinder"; see Figure 14.

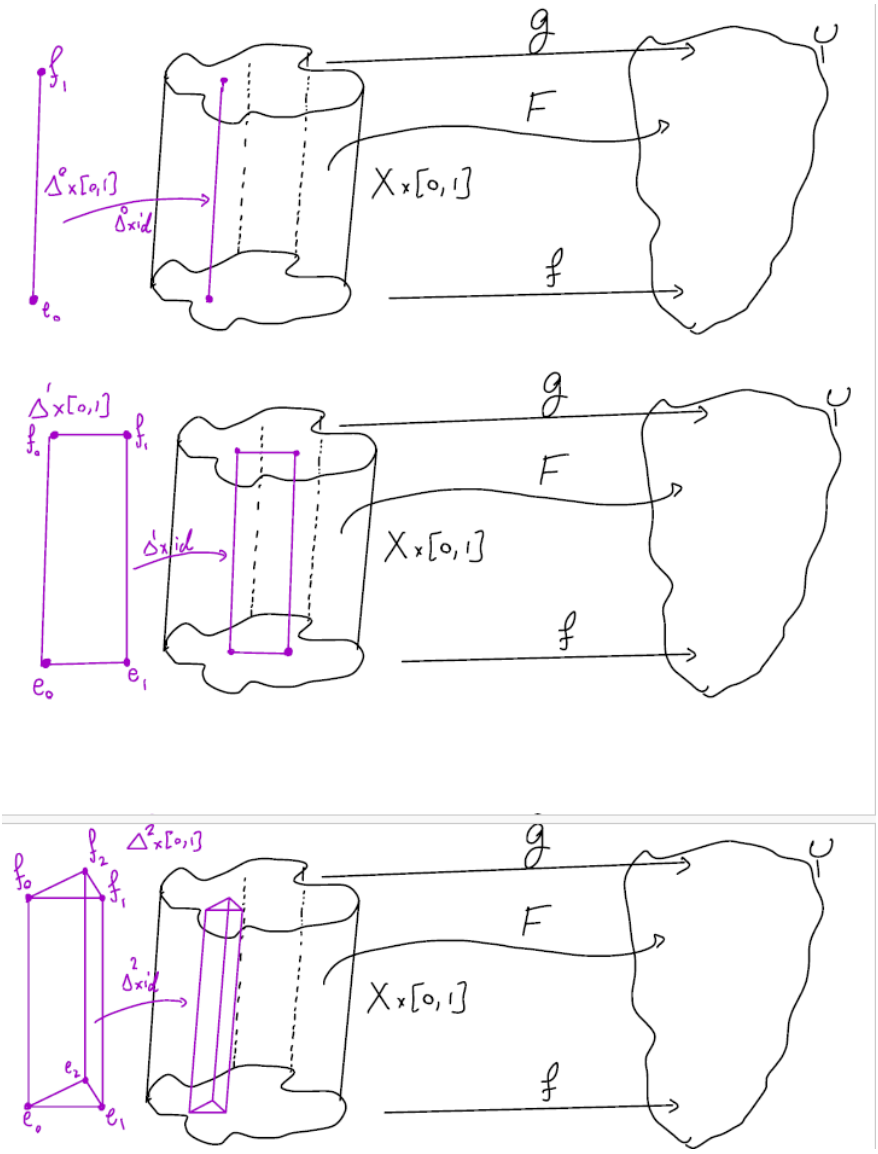


FIGURE 14. $\Delta^k \times [0,1] \subseteq X \times [0,1]$ for $k - 0, 1, 2$

Notice that $\tau \times \mathrm{id}$ inside the cylinder looks like a "prism" and we will cut this object into some number of $k+1$-simplicies in a way compatible with Definition 25. Composing these cut up $k+1$-simplicies with F yields yields $k+1$ simplicies of Y and therefore the desired p_k. I will also demonstrate this cutting process inductively for $0, 1$ and 2 simplicies of X before doing the general case.

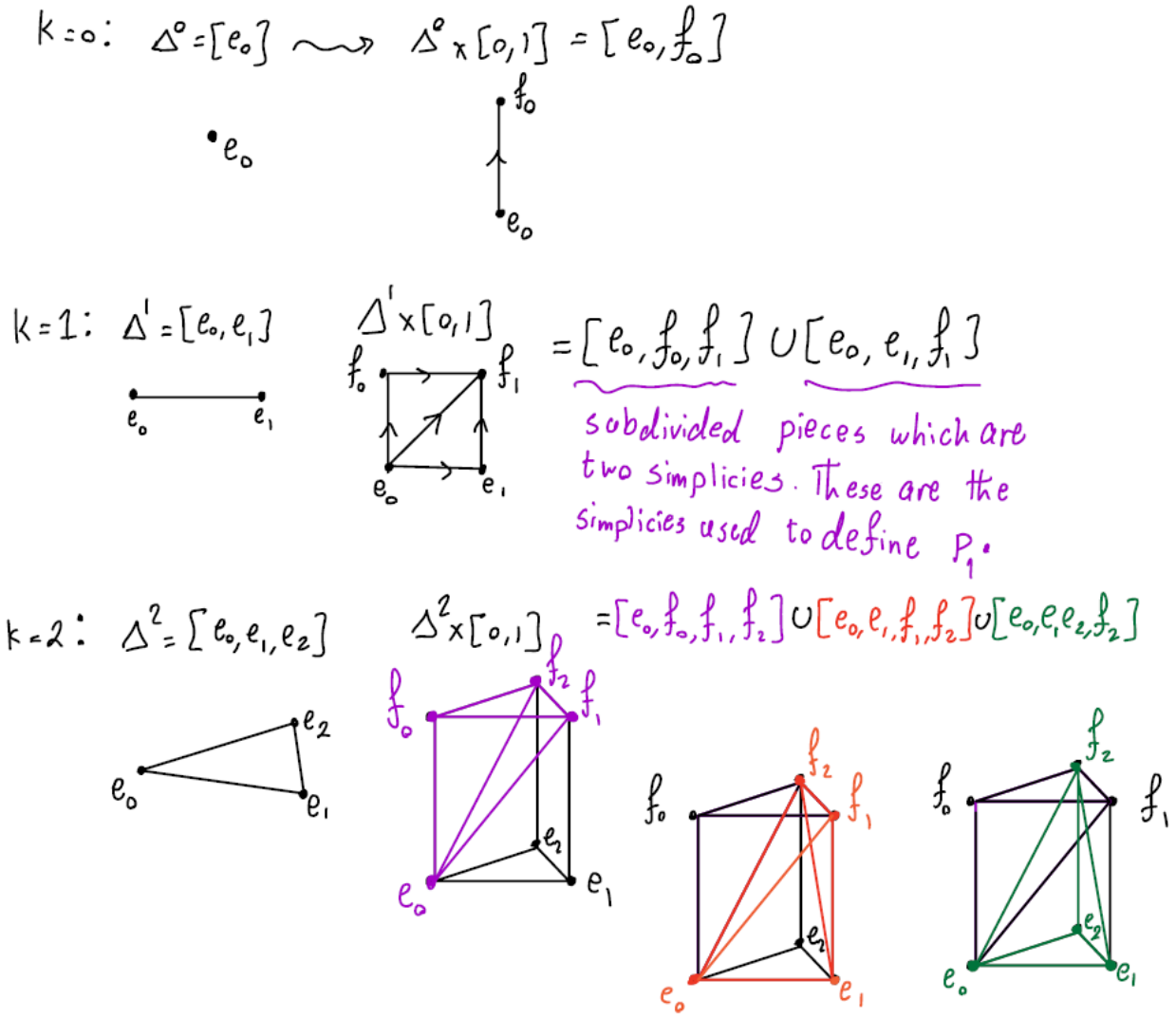


FIGURE 15. $\Delta^k \times [0,1] \subseteq X \times [0,1]$ for $k - 0, 1, 2$

Note that the color coded subdivision of $\Delta^2 \times [0,1]$ is a subdivision of the same object, it was just hard to draw in a single copy so I drew three different copies of the same object, namely $\Delta^2 \times [0,1]$ and showed each piece separately.

Moreover, $\Delta^k \times [0,1] \neq \Delta^{k+1}$ i.e. there is no linear homeomorphism taking $\Delta^k \times [0,1]$ to Δ^{k+1}. This is precisely why we subdivide $\Delta^k \times [0,1]$ into $k+1$-simplicies. The object $\Delta^k \times [0,1]$ is called a prism thus the name of p_k.

Fix a k-simplex of X say $\tau : \Delta^k \longrightarrow X$. Subdivide $\Delta^k \times [0,1]$ into the following union:

$$\Delta^k \times [0,1] = \bigcup_{i=0}^{k} [e_0, \ldots, e_i, f_i, \ldots, f_k].$$

Consider the map $F \circ (\tau \times \mathrm{id}) : \Delta^k \times [0,1] \longrightarrow Y$ and observe that $F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,f_i,\dots,f_k]}$ is indeed a $k+1$-simplex of Y i.e. $F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,f_i,\dots,f_k]} \in C_{k+1}(Y)$. Finally define

$$p_k(\tau) := \sum_{i=0}^{k} (-1)^i F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,f_i,\dots,f_k]} \tag{8.4}$$

Since we have defined p_k on the generators of $C_k(X)$, the universal property allows us to extend the map to all of $C_k(X)$ thus

$$p_k : C_k(X) \longrightarrow C_{k+1}(Y).$$

This was all just to define p_k. Next we need to show p_k is indeed a chain homotopy, that is for any k and $\tau \in C_k(X)$ the following is satisfied:

$$g_k(\tau) - f_k(\tau) = \underbrace{\delta^Y_{k+1} \circ p_k(\tau)}_{1} + \underbrace{p_{k-1} \circ \delta^X_k(\tau)}_{2}. \tag{8.5}$$

Let us now roll up our sleeves and compute 1 and 2. Let $\tau : [e_0, \dots, e_k] \to X$.

$$\begin{aligned}
1 &= \delta^Y_{k+1} \left(\sum_{i=0}^{k} (-1)^i F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,f_i,\dots,f_k]} \right) \\
&= \sum_{i=0}^{k} \left(\sum_{j=0}^{i} (-1)^{i+j} F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,\hat{e_j},\dots,e_i,f_i,\dots,f_k]} + \sum_{j=i}^{k} (-1)^{i+j+1} F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,f_i,\dots,\hat{f_j}\dots,f_k]} \right) \\
2 &= p_{k-1} \circ \delta^X_k(\tau) = p_{k-1} \left(\sum_{j=0}^{k} (-1)^j \tau|_{[e_0,\dots,\hat{e_j},\dots,e_k]} \right) = \sum_{j=0}^{k} (-1)^j \left(p_{k-1} \left(\tau|_{[e_0,\dots,\hat{e_j},\dots,e_k]} \right) \right) \\
&= \sum_{j=0}^{k} (-1)^j \left(\sum_{i=0}^{j-1} (-1)^i F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,f_j,\dots,\hat{f_i},\dots,f_k]} + \sum_{i=j+1}^{k} (-1)^{i+1} F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,\hat{e_j},\dots,e_i,f_j,\dots,f_k]} \right) \\
&= \sum_{j=0}^{k} \left(\sum_{i=0}^{j-1} (-1)^j (-1)^i F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,f_j,\dots,\hat{f_i},\dots,f_k]} + \sum_{i=j+1}^{k} (-1)^j (-1)^{i+1} F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,\hat{e_j},\dots,e_i,f_j,\dots,f_k]} \right)
\end{aligned}$$

Carefully adding the expressions 1 and 2 we get the following:

$$\delta^Y_{k+1} \circ p_k(\tau) + p_{k-1} \circ \delta^X_k(\tau) = \sum_{i=0}^{k} \left(F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,\hat{e_i},f_i,\dots,f_k]} - F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_i,\hat{f_i},\dots,f_k]} \right) \tag{8.6}$$

However, Equation (8.6) is a telescoping sum, meaning all the middle terms cancel, because of the following observation:

$$[e_0, \dots, e_i, \hat{f_i}, f_{i+1}, \dots, f_k] = [e_0, \dots, e_i, \hat{e_{i+1}}, f_{i+1}, \dots, f_k].$$

Thus Equation (8.6) turns to

$$\begin{aligned}
8.6 &= F \circ (\tau \times \mathrm{id})|_{[\hat{e_0},f_1,f_2,\dots,f_k]} - F \circ (\tau \times \mathrm{id})|_{[e_0,\dots,e_k,\hat{f_k}]} \\
&= g \circ \tau - f \circ \tau = g_k(\tau) - f_k(\tau)
\end{aligned}$$

The τ in $g \circ \tau$ and $f \circ \tau$ respectively refer to $\tau : [f_0, \dots, f_k] \to X$ and $\tau : [e_0, \dots, e_k] \to X$. Therefore $p_* : C_*(X) \longrightarrow C_{*+1}(Y)$ is a chain homotopy as desired. □

Example 28. As an immidiate example of Theorem 8.1, note that the cylinder $S^1 \times [0,1] \simeq S^1$ therefore they have the same homology groups.

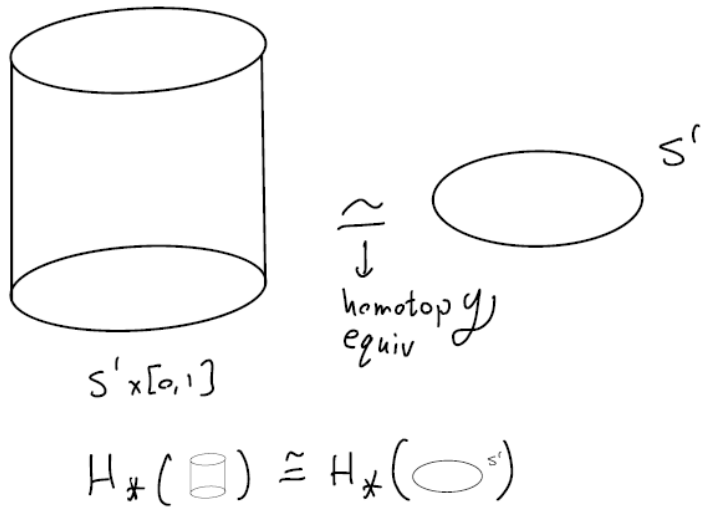


FIGURE 16. $\Delta^k \times [0,1] \subseteq X \times [0,1]$ for $k - 0, 1, 2$

9. Relative Homology

Monday February 23rd:

"Play your own way. Don't play what the public wants. You play what you want and let the public pick up on what you're doing...even if it does take them fifteen, twenty years."
-Thelonious Monk.

Recall (X, A) denotes a pair of topological whenever $A \subseteq X$ and a map of pairs $f : (X, A) \longrightarrow (Y, B)$ is the notation for $f : X \to Y$ such that $f(A) \subseteq B$. Our convention is the pair $(X, \emptyset)$ is the same as the topological space X.

Given a pair (X, A) we would like to shift our attention to those k-simplicies of X that are contained in A. Equivalentely, we would like to divide the indexing set of the cycles of X as follows:

$$C_k^{\text{sing}}(X) = \bigoplus_{\{\tau:\Delta^k \longrightarrow X\}} \mathbb{Z} \cong \underbrace{\left(\bigoplus_{\{\tau(\Delta^k) \not\subseteq A\}} \mathbb{Z} \right)}_{C_k(X,A)} \bigoplus \underbrace{\left(\bigoplus_{\{\tau(\Delta^k) \subseteq A\}} \mathbb{Z} \right)}_{C_k^{\text{sing}}(A)} \tag{9.1}$$

Notice this decomposition implies that $C_k^{\text{sing}}(A) = \left(\bigoplus_{\{\tau(\Delta^k) \subseteq A\}} \mathbb{Z} \right)$ is a free subgroup of $C_K(X)$ and Equation (9.1) implies

$$C_K(X, A) \cong \frac{C_k(X)}{C_k(A)}.$$

The group $C_K(X, A)$ is called the *relative k-chains*.

Lemma 11.

(1) $(\delta_k^X)(C_k(A)) \subset C_{k-1}(A)$
(2) $\delta_k^{(X,A)} : C_k(X,A) \longrightarrow C_{k-1}(X,A)$ *is well-defined and* $\delta_k^{(X,A)} \circ \delta_{k+1}^{(X,A)} = 0$

An immediate consequence of Lemma 11 is that $(C_*(X,A), \delta_*^{(X,A)}$ is a chain complex called *relative singular chain complex of* (X,A) and it follows that the k-th relative homology of (X,A) where $k \geq 0$ is defined as

$$H_*(X,A) = \frac{\ker(\delta_k^{(X,A)})}{\mathrm{Im}(\delta_{k+1}^{(X,A)})}$$

For example, in Figure 17, σ is not a cycle in X but it is a cycle in X/A.

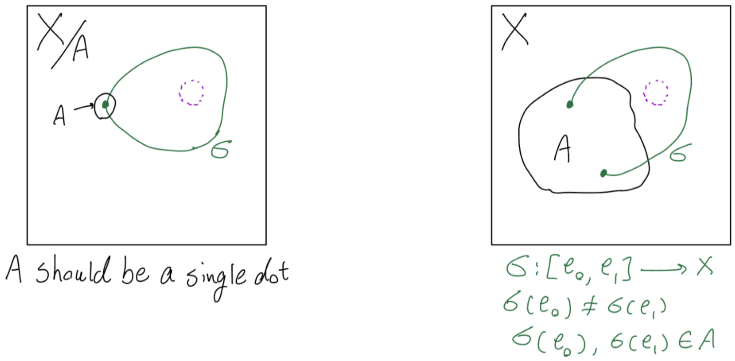


FIGURE 17. Cycles in X vs X/A

notice that $\delta_1(\sigma) \in C_0(A)$ which implies σ is a relative cycle i.e. $\delta_1^{(X,A)}(\sigma) = 0$.

Remark 8. It is crucial to note that in general:

$$H_*(X,A) \ncong \frac{H_*(X)}{H_*(A)}.$$

That is, while relative homology is related to the homologies of X and A it is **not** given by taking the quotient of said homology groups. In fact we are going to explore the following questions:

- describe the exact relationship between $H_*(A)$, $H_*(X)$ and $H_*(X,A)$,
- describe the exact relationship between $H_*(X/A)$ and $H_*(X,A)$.

Since the primary source for this course is Hatcher I will stick to the definitions and the terminology in that book. The following definitions and paragraphs are directly copied from pages 2-3 of Hatcher. We wish to define a good pair for which we need to recall what a deformation retraction is.

Definition 26 ((strong) Deformation Retraction)**.** A deformation retraction of a space X onto a subspace $A \subset X$ is a family of maps, indexed by $t \in [0,1]$, where $f_t : X \longrightarrow X$ such that

- $f_0 = \mathrm{id}_X$
- $f_1(X) = A$

- $f_t|_A = \mathrm{id}_A$ for all t

The family f_t should be continuous in the sense that the associated map $X \times I \longrightarrow X$ where $(x,t) \longrightarrow f_t(x)$ is continuous.

Recall that in Definition 23 we defined homotopic maps. A deformation retraction is a special case of a homotopic maps. In particular, a deformation retraction is a homotopy from the identity map to a retraction of X onto A which we now recall.

Definition 27 (Retraction)**.** A retraction of X onto a subspace A is a map $r : X \longrightarrow X$ such that $r(X) = A$ and $r|_A = \mathrm{id}(A)$. Equivalently, a retraction is a map $r : X \longrightarrow A$ such that $r|_A = \mathrm{id}(A)$

Definition 28 (Good Pair)**.** A pair (X, A) is called a good pair if A is a non-empty closed subspace of X such that there exists a neighborhood U of A that deformation retracts to A.

Our discussion above unpacks Definition 28 as follows: (X, A) is a good pair if A is a non-empty closed subspace of X, $A \subset U \subset X$ such that U deformation retracts onto A. This implies the maps $i : A \longrightarrow U$ and $r : U \longrightarrow A$, where r is a retraction, are homotopic. That is $r \circ i \simeq \mathrm{id}_A$ and $i \circ r \simeq \mathrm{id}_U$.

Remark 9. One might wonder if the other implication is also true: If $r : X \longrightarrow A$ is a retraction and $r \simeq \mathrm{id}_X$, is it true that X deformation retracts to A? The short answer is: it depends. Page.3 of Hatcher implicitly equivocates these notation, where he says "a deformation retraction *is* a homotopy from the identity to a retraction." While English is not my first language, I am pretty sure the word "is" means these are equivalent definitions and this is simply not true. Let me explain what is and what is not true.

Some authors defines a deformation retraction without the third requirement of Definition 26. That is, the subspace A need not be fixed by each map of the homotopy. This is called a *(weak) deformation retraction*. Using this, a weak deformation retraction *is* equivalent to a homotopy from the identity to a retract. (This is an exercise in notation manipulation.) This is **not** the definition of deformation hatcher is using on page 2.

If we start with a retraction $r : X \longleftarrow A$ and consider a homotopy between $i \circ r$ and id_X, where $i : A \hookrightarrow X$, apriori there is no reason to believe this homotopy is a (strong) deformation retraction; the maps of this homotopy at $t \neq 0, 1$ need not fix A through out the process which is a requirement of a (strong) deformation retraction.

On the other hand, Munkres in [Mun84] makes a similar claim as Hatcher does (page 108-109)! And given that Munkres is a far more careful writer than Hatcher, I was very confused at first. But in my understanding, Munkres is applying this to the setting of CW complexes and in that setting these definitions are equivalent, albeit that is not obvious either and requires equivalence of homotopies.

Example 29. An important example of a good pair is $(D^m, \delta D^m)$, the m-dimensional disk and its boundary δD^m. Recall that

$$D^m = \{(x_1, \ldots, x_m) \mid \sum x_i \leq 1\}$$
$$\delta D^m = \{(x_1, \ldots, x_m) \mid \sum x_i = 1\} = S^{m-1}$$

Following Definition 28, notice that $U_\epsilon = \{(x_1, \ldots, x_m) \mid 1 - \epsilon < ||\vec{x}|| \leq 1\}$ is clearly open, contains δD^m and it deformation retracts onto δD^m.

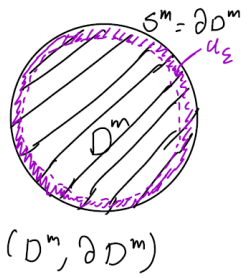


FIGURE 18. $(D^m, \delta D^m)$ forms a good pair

Example 30. [Two non-examples] In your homework you will show that neither of the following examples constitutes a good pair.

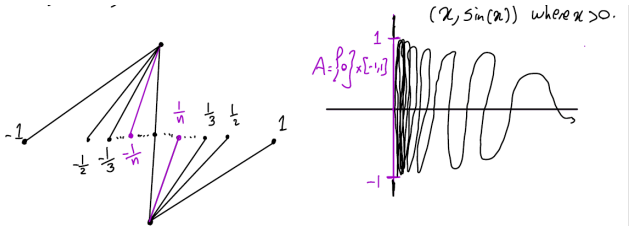


FIGURE 19. Two non-examples of a good pair

The left hand picture is the union of all line segments joining the point $(0, -1)$ to each point of the set $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ and all lines segments joining $(0, 1)$ to each point of the set $\{-\frac{1}{n}\}_{n\in\mathbb{N}}$ the right hand curve is topologist's sin curve, a better picture of which can be found on wikipedia.

Denote left hand side curve with X and let $a_0 = (0, 0)$. Moreover, let X' denote the topologist sin curve and $A = 0 \times [-1.1]$ be the vertical line segment from -1 to 1. In your homework, you will justify why $(X, \{a_0\})$ and (X', A) are not good pairs.

A main reason for introducing the long exact sequence in homology is to compute relative homology for good pairs. As a motivating example, is it true that $H_*(D^m, \delta D^m) \cong H_*(\underbrace{D^m/\delta D^m}_{\cong S^m})$?

Theorem 9.1 (Relative homology for good pairs)**.** *If (X, A) is a good pair then*

$$H_*(X, A) \cong \tilde{H}(X/A).$$

Wednesday February 25th:

"Drink your coffee, embrace the silence, do not take people seriously, do not take life upon yourself, do not exaggerate your emotions, and do not please anyone against your will."
-Mahmoud Darwish.

In order to relate the three homology groups $H_*(A)$, $H_*(X)$ and $H_*(X, A)$ we need to introduce and study exact sequences.

Definition 29 (Exact Sequences)**.** A sequence of abelian groups and homomomorphisms

$$\cdots \xrightarrow{a_{k+2}} A_{k+1} \xrightarrow{a_{k+1}} A_k \xrightarrow{a_k} A_{k-1} \xrightarrow{a_{k-1}} A_{k-2} \xrightarrow{a_{k-2}} \cdots$$

is called exact if and only if for all k, including when $k < 0$, we have

$$\mathrm{Im}(a_k) = \ker(a_{k+1}).$$

Note that all homology groups of an exact sequence are 0. Therefore the existence of homology is in some sense measuring how far a complex is from being exact.

The following is straightforward to show and I will not assign it as homework. However, if you are not comfortable working with short exact sequences you should work them out.

Lemma 12.

- $0 \longrightarrow A \xrightarrow{\alpha} B$ *is exact if and only if* α *is injective, also called a monomorphism.*
- $A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if* α *is surjective, also called an epimorphism.*
- $0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$ *is exact if and only if* α *is an isomorphism.*
- $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ *is exact if and only if* α *is a monomorphism,* β *is an epimorphism and* $\ker(\beta) = \mathrm{Im}(\alpha)$ *if and only if* $C \cong \dfrac{B}{\ker \beta} \cong \dfrac{B}{\mathrm{Im}(\alpha)} \cong \dfrac{B}{A}$

Definition 30 (Short Exact Sequence of Chain Complexes)**.** Consider chain complexes (A_*, a_*) (B_*, b_*), (C_*, c_*) and chain homomorphisms $\alpha_* : (A_*, a_*) \to (B_*, b_*)$ and $\beta_* : (B_*, b_*) \to (C_*, c_*)$. The sequence

$$0 \longrightarrow (A_*, a_*) \xrightarrow{\alpha_*} (B_*, b_*) \xrightarrow{\beta_*} (C_*, c_*) \longrightarrow 0$$

is called a short exact sequence of chain complexes if and only if for any k, each square of the following diagram commutes and each of its rows forms a short exact sequence.

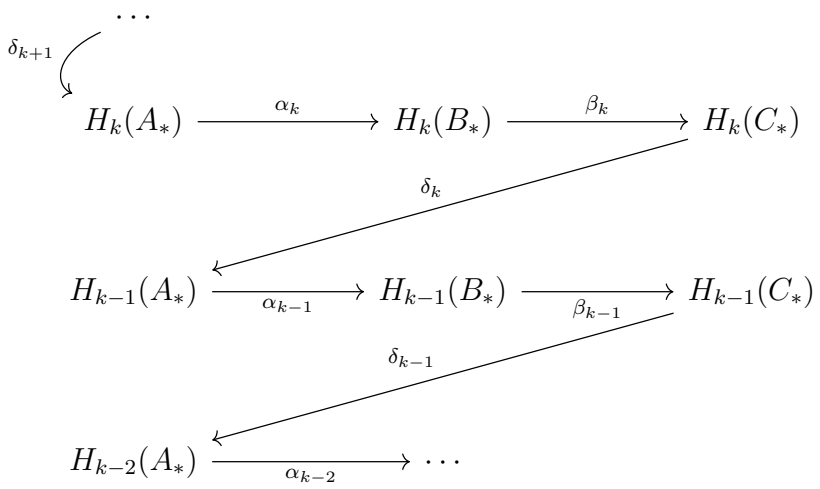
$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle a_{k+2}} & & \downarrow{\scriptstyle b_{k+2}} & & \downarrow{\scriptstyle c_{k+2}} & & \\
0 & \longrightarrow & A_{k+1} & \xrightarrow{\alpha_{k+1}} & B_{k+1} & \xrightarrow{\beta_{k+1}} & C_{k+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_{k+1}} & & \downarrow{\scriptstyle b_{k+1}} & & \downarrow{\scriptstyle c_{k+1}} & & \\
0 & \longrightarrow & A_k & \xrightarrow{\alpha_k} & B_k & \xrightarrow{\beta_k} & C_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_k} & & \downarrow{\scriptstyle b_k} & & \downarrow{\scriptstyle c_k} & & \\
0 & \longrightarrow & A_{k-1} & \xrightarrow{\alpha_{k-1}} & B_{k-1} & \xrightarrow{\beta_{k-1}} & C_{k-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_{k-1}} & & \downarrow{\scriptstyle b_{k-1}} & & \downarrow{\scriptstyle c_{k-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

A short note on notation: in Definition 30, the maps α_* and β_* induce maps on homology as in Lemma 5. Recall this map is to apply α_* or β_* to a representative of an equivalence class of a cycle. For ease of notation, the statement of Theorem 9.2 uses α_* to also denote the induced map on homology group (even though it should technically be $H_*(\alpha_*)$.)

Theorem 9.2. *Given a short exact sequence of chain complexes yields a long exact sequence in homology. This means for each k there exists a map $\delta_k : H_k(C_*) \longrightarrow H_{k-1}(A_*)$ satisfying*

(1) $\mathrm{Im}(\delta_k) = \ker(\alpha_{k-1})$,
(2) $\mathrm{Im}(\alpha_{k-1}) = \ker(\beta_{k-1})$ *(This does not depend on δ)*
(3) $\mathrm{Im}(\beta_{k-1}) = \ker(\delta_{k-1})$.

The diagram for this long exact sequence looks like



The map δ_* is called the "connecting homomorphism."

We will construct the map δ during the course of the proof.

Friday February 27th:

"Honesty always gets my attention. Not particularly someone who is honest to me, but someone who is honest with themselves."

-Heath Ledger.

Proof. One should keep an eye out on the diagram in Definition 30. Let $[\gamma] \in H_k(C_*)$. Fix a representative γ for this class i.e. $[\gamma] = \gamma + \mathrm{Im}(c_{k+1})$ where $\gamma \in \ker(c_k)$. We will do a diagram chase at the level of chains i.e. in the diagram in Definition 30, to define a map δ_k. Then we show δ_k induces a well-defined map on $H_k(C_*)$. Since $\gamma \in \ker(c_k) \subset C_k$ and β_k is surjective, then there exists $\zeta \in B_k$ such that $\beta_k(\zeta) = \gamma$. Note that $b_k(\zeta) \in B_{k-1}$ so it makes sense to apply β_{k-1} to $b_k(\zeta)$. The commutativity of the diagram for the first equality we get

$$\beta_{k-1}(b_k(\zeta)) = c_k(\underbrace{\beta_k(\zeta)}_{\gamma \in \ker(c_k)}) = 0.$$

Therefore $b_k(\zeta) \in \ker(\beta_{k-1}) = \mathrm{Im}(\alpha_{k-1})$. Since α_{k-1} is injective, there exists a *unique* $\psi \in A_{k-1}$ such that $\alpha_{k-1}(\psi) = b_k(\zeta)$. We now define $\delta_k(\gamma) := \psi$ which we have arrived at through this process. We now need to make sure δ_k induces a well-defined map on homology i.e. check that the map

$$\delta_k : H_k(C) = \frac{\ker(c_k)}{\mathrm{Im}(c_{k+1})} \longrightarrow H_{k-1}(A) = \frac{\ker(a_{k-1})}{\mathrm{Im}(a_k)}$$
$$\delta_k(\gamma + \mathrm{Im}(c_{k+1})) = \delta_k(\gamma) + \mathrm{Im}(a_k) = \psi + \mathrm{Im}(a_k)$$

is well-defined. This amounts to showing δ_k lands in the right space and two different representatives of the same equivalence class are mapped to the same equivalence class under δ_k.

(1) $\delta_k(\ker(c_k)) \subset \ker(a_{k-1})$
(2) If $[\gamma] = [\gamma']$ i.e. $\gamma - \gamma' \in \mathrm{Im}(c_{k+1})$ and $\delta_k(\gamma) = \psi$, $\delta_k(\gamma') = \psi'$ then $\psi - \psi' \in \mathrm{Im}(a_k)$.

Proof of 1: Let $\gamma \in \ker(c_k)$ and $\psi = \delta_k(\gamma)$. We want to show $\psi \in \ker(a_{k-1})$ i.e. $a_{k-1}(\psi) = 0$. The commutativity of the diagram in Definition 30 shows

$$\alpha_{k-2}(a_{k-1}(\psi)) = b_{k-1}(\alpha_{k-1}(\psi)) = 0$$

where the final equality follows because $\alpha_{k-1}(\psi) = b_k(\zeta)$ from the construction of δ_k and $b_{k-1} \circ b_k = 0$. Since α_{k-2} is injective we get $a_{k-1}(\psi) = 0$ i.e. $\psi \in \ker(a_{k-1})$. as desired.
Proof of 2: Assume the set up of 2. Since $\gamma - \gamma' \in \mathrm{Im}(c_{k+1})$, there exists an element $\epsilon \in C_{k+1}$ such that $\gamma - \gamma' = c_{k+1}(\epsilon)$. Since β_{k+1} is surjective, there exists $\mu \in B_{k+1}$ such that $\beta_{k+1}(\mu) = \epsilon$. Hold on to this element μ! Recall that $\gamma, \gamma' \in C_k$ and β_k is surjective therefore there exists $\zeta.\zeta'$ such that $\beta_k(\zeta) = \gamma$ and $\beta_k(\zeta') = \gamma'$. Moreover, as we discussed in the first part of the proof, ζ and ζ' satisfy the property that $\alpha_{k-1}(\psi) = b_k(\zeta)$ and $\alpha_{k-1}(\psi') = b_k(\zeta')$.

Notice that ζ, ζ' and $b_{k+1}(\mu)$ all live in B_k therefore $\zeta - \zeta' - b_{k+1}(\mu) \in B_k$. Next, we will show $\zeta - \zeta' - b_{k+1}(\mu) \in \ker(\beta_k)$.

$$\beta_k(\zeta - \zeta' - b_{k+1}(\mu)) = \beta_k(\zeta) - \beta_k(\zeta') - \beta_k(b_{k+1}(\mu)) = \gamma - \gamma' - c_{k+1}(\underbrace{\beta_{k+1}(\mu)}_{\epsilon}) = 0$$

Therefore $\zeta - \zeta' - b_{k+1}(\mu) \in \ker(\beta_k) = \mathrm{Im}(\alpha_k)$. Therefore $\alpha_k(a) = \zeta - \zeta' - b_{k+1}(\mu)$ for some $a \in A_k$. Finally, note that

$$\alpha_{k-1}(a_k(a)) = b_k(\alpha_k(a)) = b_k(\zeta - \zeta' - b_{k+1}(\mu)) = b_k(\zeta - \zeta') - \underbrace{b_k(b_{k+1}(\mu))}_{0} = \alpha_{k-1}(\psi - \psi').$$

Since α_{k-1} is injective we get that $a_k(a) = \psi - \psi'$, that is $\psi - \psi' \in \mathrm{Im}(a_k)$ as desired. One still needs to show that δ_k is indeed a homomorphism but I claim that is clear since all the maps involved are homomorphisms. Finally, we need to show exactness in three spots as described in the statement of the theorem but I will leave that as homework. □

Monday March 2nd:

"If we opened people up, we'd find landscapes."

-Agnes Varda.

Remark 10. Given a good pair (X, A) and each $k \geq 0$ we have the s.e.s

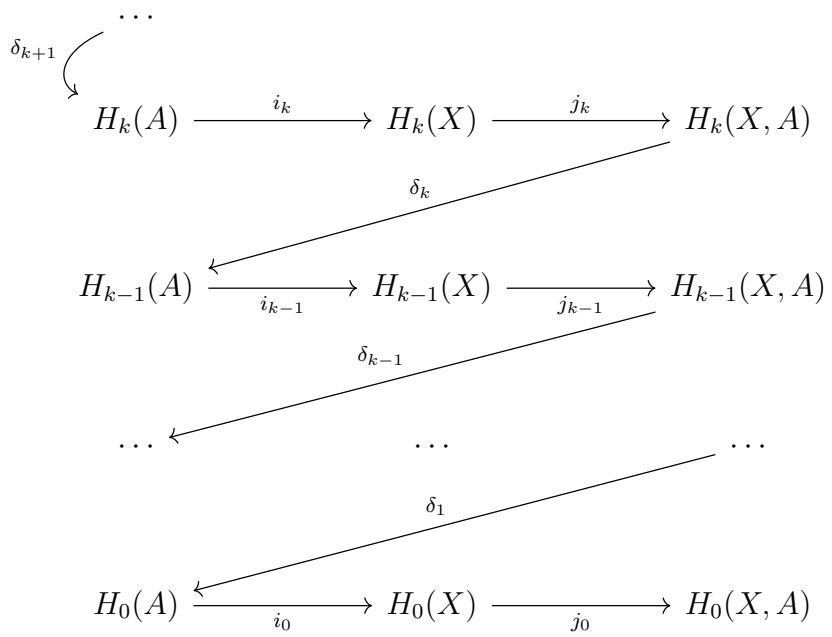
$$0 \longrightarrow C_k(A) \xrightarrow{i_k} C_k(X) \xrightarrow{j_k} \underbrace{\frac{C_k(X)}{C_k(A)}}_{C_k(X,A)} \longrightarrow 0$$

where i_k is induced from the inclusion map $i : A \hookrightarrow X$; given $\tau : \Delta^K \longrightarrow A$ define $i_k(\tau) = (i \circ \tau) : \Delta^k \longrightarrow X$, and j_k is the projection map. This yields a short exact sequence of

complexes:

$$0 \longrightarrow C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{j_*} \underbrace{\frac{C_*(X)}{C_*(A)}}_{C_*(X,A)} \longrightarrow 0.$$

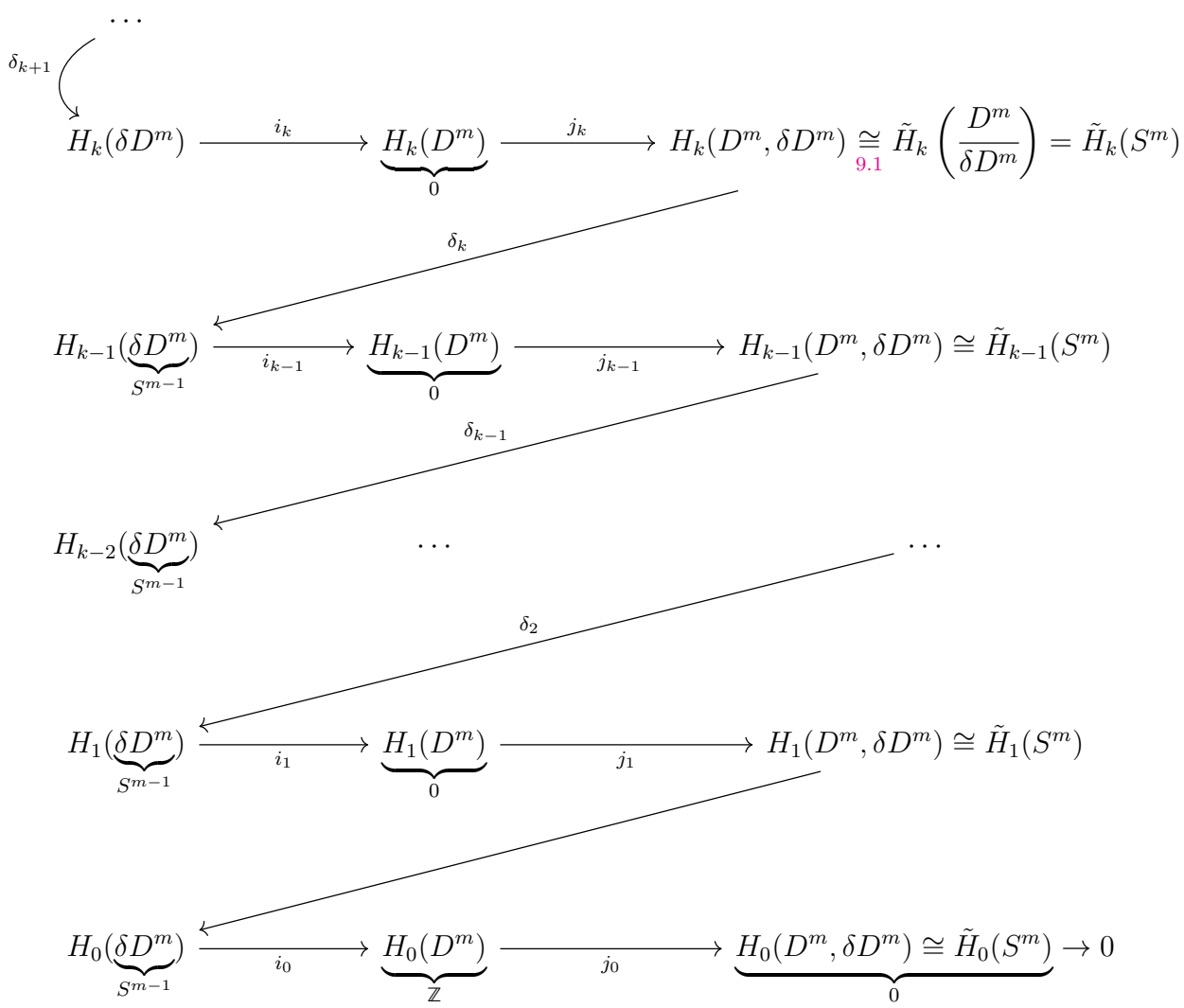
Applying Theorem 9.2 we obtain the long exact sequence in homology which looks like



Note that an identical sequence exists where Homology groups are replaced with relative homology groups appropriately.

Example 31. Fix $m \geq 1$. Let us now apply Theorem 9.2, via Remark 10 to the case of $(D^m, \delta D^m)$, which we have shown in Example 29 is a good pair. Our goal is to compute $H_k(D^m, \delta D^m)$, which, by Theorem 9.2 is isomorphic to $\tilde{H}_k(D^m/\delta D^m) \cong \tilde{H}_k(S^m)$.

The long exact sequence in homology we get will look like



In degree 0 we get $H_0(D^m, \delta D^m) \cong \tilde{H}_0(S^m)$. Let us now justify a slightly more general fact which we have not discussed before: for any $m \geq 1$, we have

$$H_k(S^m) = \begin{cases} \mathbb{Z} & \text{if } k = 0, m \\ 0 & \text{otherwise} \end{cases} \tag{9.2}$$

To see this, note that S^m has exactly one path connected component thus its zeroth homology group is $\mathbb{Z}$ by Corollary 3. Moreover, we can attach a single m-simplex in degree m (think of S^1 or S^2) and that justifies why $H_m(S^m) \cong \mathbb{Z}$. This implies the *reduced* homology S^m in degree 0 is zero and justify the penultimate zero in the last row.

Let's analyze $H_1(D^m, \delta D^m)$ and the last part of this long exact sequence further:

$$0 \to H_1(D^m, \delta D^m) \cong \tilde{H}_1(S^m) \xrightarrow{\delta_1} \underbrace{H_0(S^{m-1})}_{\tilde{H}_0(S^{m-1}) \oplus \mathbb{Z}} \xrightarrow{i_0} \underbrace{H_0(D^m)}_{\cong \mathbb{Z}} \to 0$$

By exactness i_0 is surjective therefore $\dfrac{\tilde{H}_0(S^{m-1}) \oplus \mathbb{Z}}{\ker(i_0)} \cong \mathbb{Z}$ thus $\ker(i_0) = \tilde{H}_0(S^{m-1})$. Using exactness again, which implies δ_1 is injective we get

$$\tilde{H}_1(S^m) = \mathrm{Im}(\delta_1) = \ker(i_0) = \tilde{H}_0(S^{m-1}) \text{ for any } m \geq 1. \tag{9.3}$$

Moreover, the exactness of the long exact sequence and the fact that $H_k(D^m) = 0$ for all k implies

$$\tilde{H}_k(S^m) \cong H_k(D^m, \delta D^m) \cong H_{k-1}(S^{m-1}) \text{ for all } k \geq 2.$$

To finish the computation, recall that reduced homology and homology are the same in all degrees except zero and that m and k were arbitrary. Hence we end up with a sequence of isomorphisms as follows:

$$\begin{aligned}\tilde{H}_k(S^m) &\cong H_k(D^m, \delta D^m) \cong H_{k-1}(S^{m-1}) \cong \tilde{H}_{k-1}(S^{m-1}) \cong H_{k-1}(D^{m-1}, \delta D^{m-1}) \cong H_{k-2}(S^{m-2}) \\ &\cong \tilde{H}_{k-2}(S^{m-2}) \cong H_{k-2}(D^{m-2}, \delta D^{m-2}) \cong H_{k-3}(S^{m-3}) \cong \cdots \cong \underbrace{\tilde{H}_1(S^{m-(k-1)})}_{(*)} \cong H_1(D^{m-(k-1)}, \delta D^{m-(k-1)})\end{aligned} \tag{9.4}$$

First, note that the exact value of m determines how much of the sequence of homologies above we will actually use. Second, applying Equation (9.3) to the homology group $(*)$, we get

$$\tilde{H}_1(S^{m-k+1}) \cong \tilde{H}_0(S^{m-k})$$

Finally, combining Equation (9.4) and the previous isomorphism we get

$$H_k(D^m, \delta D^m) \cong \tilde{H}_k(S^m) \cong \tilde{H}_0(S^{m-k}).$$

If $m < k$, then as we discussed earlier in this example, there are no k-simplicies so the middle group $\tilde{H}_k(S^m)$ is zero already and that is ultimately what we want to compute. If $m \geq k$, then by what we discussed earleir in this example, S^{m-k} has one connected component, therefore its homology in degree 0 in $\mathbb{Z}$ therefore its reduced homology in 0 thus $\tilde{H}_0(S^{m-k}) = 0$ for $m > k$ and $m < k$. The final computation is when $m = k$. In this case, $S^0 = \{-1, 1\}$ thus $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$ which implies $\tilde{H}_0(S^0) = \mathbb{Z}$. Putting all of this together we get:

$$H_k(D^m, \delta D^m) \cong \tilde{H}_k(S^m) \cong \tilde{H}_0(S^{m-k}) = \begin{cases} \mathbb{Z} & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}$$

10. Mayer-Vietoris long exact sequence

Wednesday March 4th:

"If you don't become the ocean, you'll be seasick every day."

-Leonard Cohen.

The long exact sequence in homology Theorem 9.2 is an abstract construction. In the previous section, we applied this construction to a (good) pair (X, A) and computed relative Homology. In this section, we wish to apply Theorem 9.2 to the situation where we have a cover of X. Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X. Recall this means $U_\alpha \subset X$ is open and $X = \bigcup U_\alpha$.

Remark 11. Fix a space X and an open cover $\mathcal{U}$ of X. Consider a chain complex structure on X. Fix $k \geq 0$ and let $C_k^{\mathcal{U}}(X)$ be the subgroup of $C_k(X)$ defined as

$$C_k^{\mathcal{U}}(X) = \left\{ \sum k_i \tau_i \mid \tau_i : \Delta^k \longrightarrow U_\alpha \subset X \text{ for some } \alpha \right\}$$

Further define

$$\delta_k^{\mathcal{U}} := \delta_k|_{C_k^{\mathcal{U}}(X)}$$
$$\delta_k^{\mathcal{U}} : C_k^{\mathcal{U}}(X) \longrightarrow C_{k-1}^{\mathcal{U}}(X)$$

That $\delta_k^{\mathcal{U}}$ has the correct target is a straightforward observation, if $\Delta^k \subset U_\alpha$ for some α, then all of its faces are also contained in U_α. Therefore $C_*^{\mathcal{U}} = (C_*^{\mathcal{U}}(X), \delta_*^{\mathcal{U}})$ is a chain complex.

By our construction, $C_k^{\mathcal{U}}(X) \overset{i}{\hookrightarrow} C_k(X)$ for all k thus $i_* : C_*^{\mathcal{U}}(X) \overset{i}{\hookrightarrow} C_*(X)$ is a chain homomorphism and it induces a map on homology:

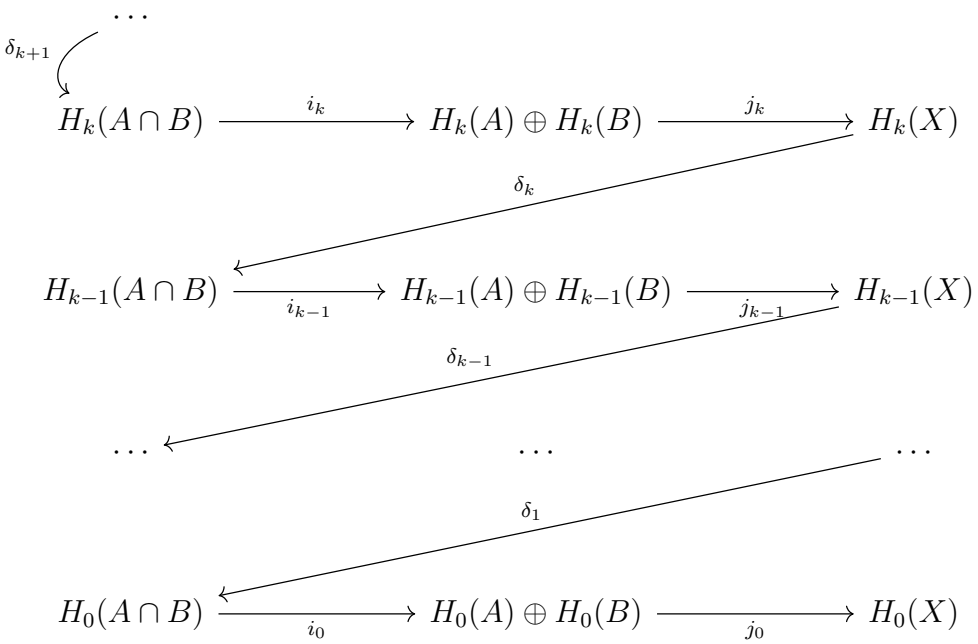
$$i_* : H_*^{\mathcal{U}}(X) \longrightarrow H_*(X). \tag{10.1}$$

It turns out that the map in Equation (10.1) is indeed an isomorphism. The precise statement is the following proposition 2.21 from Hatcher. The proof is rather long and technical, see Hatcher.

Proposition 1. *The inclusion $C_k^{\mathcal{U}}(X) \overset{i}{\hookrightarrow} C_k(X)$ is a chain homotopy equivalence, that is, there is a chain map $\rho : C_n(X) \longrightarrow C_n^{\mathcal{U}}(X)$ such that $i \circ \rho$ and $\rho \circ i$ are chain homotopic to the identity. Hence the induced map on Homology of i, namely Equation (10.1) is an isomorphism and we get*

$$H_*^{\mathcal{U}}(X) \cong H_*(X).$$

Theorem 10.1 (Mayer-Vietoris sequence)**.** *Let $A, B \subset X$ and $X = Int(A) \cup Int(B)$. There is a long exact sequence in homology called the Mayer-Vietoris long exact sequence*



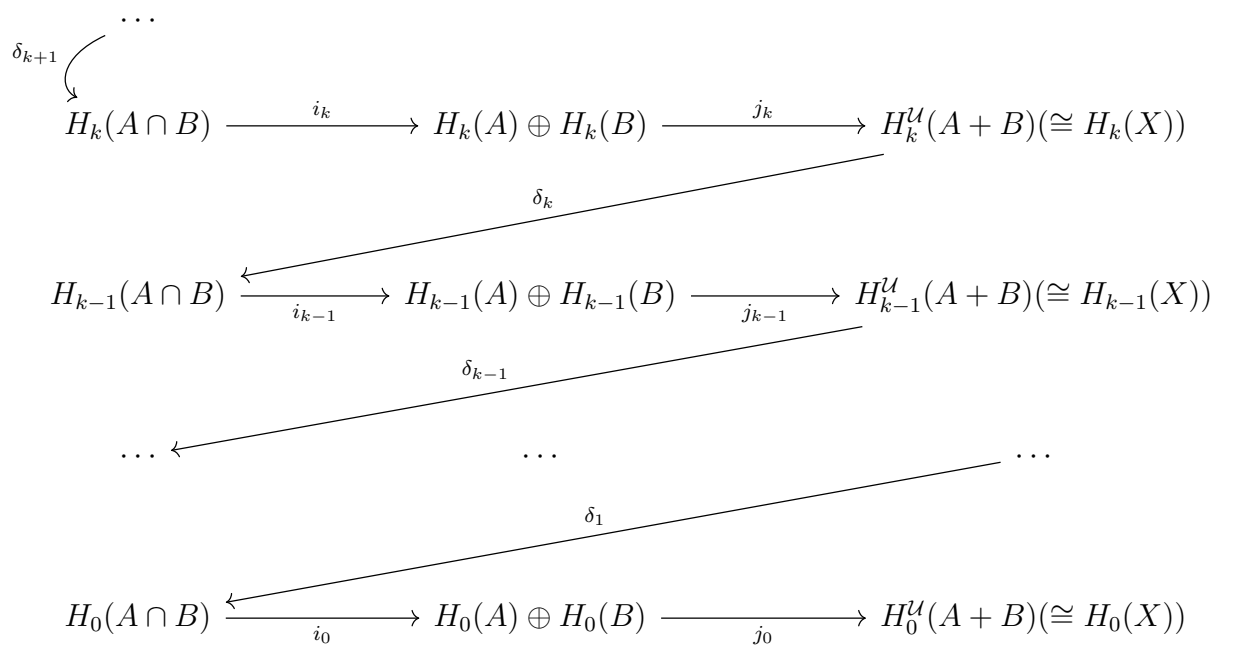
Proof. What are we proving here? We want to show there exists a long exact sequence in Homology as describe in the statement of the theorem. First we will replace homology groups

of X with something they are isomorphic to. Notice the interiors of A and B form an open cover of X by assumption. Let us write the chain complex in Remark 11 in our setting.

$$\begin{aligned} C_k^{\mathcal{U}}(A+B) &= \{k_a\tau_a + k_b\tau_b\} \text{ where} \\ \tau_a &: \Delta^k \longleftrightarrow \text{Int}(A) \subset X \\ \tau_b &: \Delta^k \longleftrightarrow \text{Int}(B) \subset X \\ \delta_k^{\mathcal{U}} &:= \delta_k|_{C_k^{\mathcal{U}}(A+B)} \\ \delta_k^{\mathcal{U}} &: C_k^{\mathcal{U}}(A+B) \longrightarrow C_{k-1}^{\mathcal{U}}(A+B) \end{aligned}$$

In words, $C_k^{\mathcal{U}}(A+B)$ is those chains that are sums of chains in A and B. There is a little abuse of notation going on here, if I wanted to keep the notation consistent with Remark 11 I should have written $C_k^{\mathcal{U}}(X)$ with the understanding that the cover consists of the two interiors. However, $C_k^{\mathcal{U}}(A+B)$ is more descriptive when it comes to defininig map. The upshot is $C_k^{\mathcal{U}}(A+B) = C_k^{\mathcal{U}}(X)$.

Therefore $C_*^{\mathcal{U}} = (C_k^{\mathcal{U}}(A+B), \delta_k^{\mathcal{U}})$ is a chain complex. By Proposition 1, the induced map on homology from the inclusion $i_* : C_k^{\mathcal{U}}(A+B) \hookrightarrow C_k(A+B)$ is an isomorphism therefore $H_*(X) \cong H_*^{\mathcal{U}}(A+B)$. Therefore it suffices to show there exists a long exact sequence homology looking like



To show such a long exact sequence in homology exists, it suffices to show there exists a short exact sequence of complexes

$$0 \to C_*(A\cap B) \longrightarrow C_*(A)\oplus C_*(B) \longrightarrow C_*(A+B) \to 0$$

and the result follows by Theorem 9.2. Now, technically I never defined what the complexes $C_*(A)\oplus C_*(B)$ and $C_*(A\cap B)$ are but I claim it is straightforward to compute the chain groups and maps in these complexes. Finally, what is left to show is for any k there exists a short exact sequence of the form

$$0 \to C_k(A\cap B) \xrightarrow{\phi_k} C_k(A)\oplus C_k(B) \xrightarrow{\psi_k} C_k(A+B) \to 0. \tag{10.2}$$

Next, we will define the maps ϕ and ψ, and show the exactness of the sequence in Equation (10.2). Note the inclusion map $i_A : A \cap B \hookrightarrow A$ induces a homomorphism $(i_a)_k : C_k(A \cap B) \hookrightarrow C_k(A)$. Same is true for i_B. We now define

$$\begin{aligned} \phi_k : C_k(A \cap B) &\longrightarrow C_k(A) \oplus C_k(B), \\ \phi_k(c) &= ((i_A)_*(c), -(i_B)_*(c)), \\ \psi_k : C_k(A) \oplus C_k(B) &\longrightarrow C_k(A + B), \\ \psi_k((a, b)) &= a + b \end{aligned}$$

Firstly we show ϕ_k is a monomorphism. Let $\tau : \Delta^k \to A \cap B$ and observe

$$\begin{aligned} & \phi_k(c) = 0 = ((i_A)_*(c), -(i_B)_*(c)) \\ \iff & (i_A)_*(c) = 0 \\ \iff & i_A \circ c = 0 \\ \iff & c = 0 \\ \iff & \ker(\phi_k) = 0 \end{aligned}$$

Next, we want $\ker(\psi_k) = \mathrm{Im}(\phi_k)$. For any $\tau \in A \cap B$ we have

$$(\psi_k \circ \phi_k)(\tau) = \psi((i_a)_*(\tau), -(i_B)_*(\tau)) = (i_a)_*(\tau) - (i_B)_*(\tau) = 0$$

where the last equality follows because $\tau \in A \cap B$ therefore its inclusion in A and B does not change it. thus $\mathrm{Im}(\phi_k) \subset \ker(\psi_k)$. Let $(a, b) \in \ker(\psi_k)$.

$$\begin{aligned} \psi_k((a, b)) = 0 = a + b \iff & \underbrace{a}_{\in C_k(A)} = \underbrace{-b}_{\in C_k(B)} \\ \Rightarrow & a = (i_A)_*(a) \ ; \ (i_B)_*(a) = -b \\ \Rightarrow & \phi_k(a, -a) = (a, b) \\ \Rightarrow & (a, b) \in \mathrm{Im}(\phi_k) \end{aligned}$$

To see ψ_k is an epimorphism let $\tau_a + \tau_b$, where $\tau_a : \Delta^k \to A$ and $\tau_b : \Delta^k \to B$. The preimage of this under ψ_k is $(\tau_a, -\tau_b)$. □

Friday March 6th:

"No one expected me. Everything awaited me."

-Patti Smith.

Corollary 6. *We can replace the homology groups in the long exact sequence of Theorem 10.1 with $\tilde{H}_*(\cdot)$, the reduced homology groups.*

Example 32. [Klein Bottle using Mayer-Vietoris Sequence] Recall the "flattened out" version of the klein bottle in Figure 20. To put ourselves in the hypothesis of Theorem 10.1 let $A = \overset{\circ}{D}{}^2$, the open disk and $B = K \backslash \{D^2_\epsilon\}$ for some $\epsilon > 0$. That is, B is the Klein bottle with a hole in the middle. Note that A is contractible to a point (if you are not convinced of this then try convincing yourself by recalling the definition.) Moreover, $B \simeq S^1 \vee S^1$ where $\vee$ denotes the Wedge Sum. This means gluing two simplicial complexes at a point. More precisely, taking the disjoint union of X and Y and taking the quotient where two points are identified. see Figure 20. Moreover, $A \cap B \simeq S^1$.

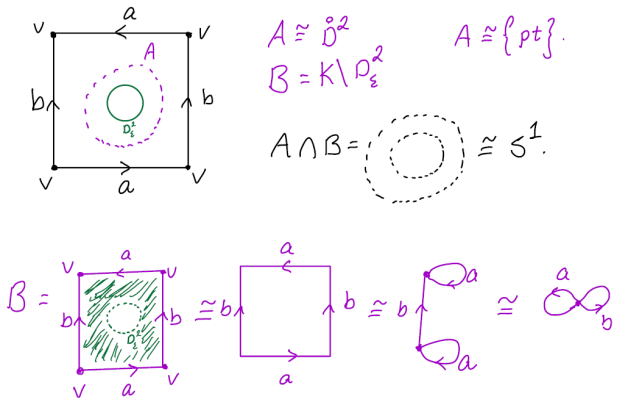


FIGURE 20. Computing the homology of the Klein bottle via M-V sequence

Putting this information into Mayer-Vietoris sequence for reduced homology we get

$$
\begin{array}{ccccc}
\cdots & & & & \\
\delta_{k+1}\downarrow & & & & \\
\underbrace{H_k(A\cap B\cong S^1)}_{0} & \xrightarrow{i_k} & H_k(A)\oplus H_k(B) & \xrightarrow{j_k} & H_k(K) \\
 & \searrow^{\delta_k} & & & \\
\underbrace{H_{k-1}(A\cap B\cong S^1)}_{0} & \xrightarrow[i_{k-1}]{} & H_{k-1}(A)\oplus H_{k-1}(B) & \xrightarrow[j_{k-1}]{} & H_{k-1}(K) \\
 & \searrow^{\delta_{k-1}} & & & \\
\cdots & & \cdots & & \cdots \\
 & \searrow^{\delta_1} & & & \\
H_0(A\cap B)\cong S^1 & \xrightarrow[i_0]{} & H_0(A)\oplus H_0(B) & \xrightarrow[j_0]{} & H_0(K)
\end{array}
$$

Using the above sequence, and the fact that $\tilde{H}_k(A) = 0$ for all k since A is contractible, for any $n > 1$ we get

$$\tilde{H}_n(K) \cong \tilde{H}_n(A) \oplus \tilde{H}_n(B) = \tilde{H}_n(S^1 \vee S^1). \tag{10.3}$$

To compute $\tilde{H}_n(S^1 \vee S^1)$ we use M-V again! and break up the space as described in the following Figure 21.

Once again, note that $A \cap B$ is contractible thus its reduced homology is zero. Plugging this into the M-V sequence, we get

$$\tilde{H}_n(S^1 \vee S^1) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

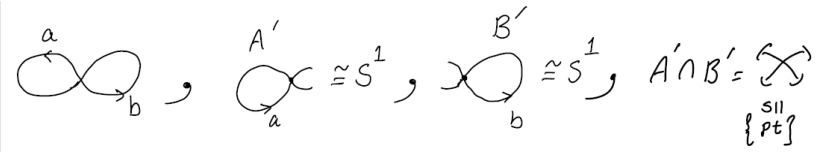


FIGURE 21. Computing the homology of the Klein bottle via M-V sequence

from Equation (10.3) we get

$$\tilde{H}_n(K) = 0 \text{ for } n > 2.$$

Finally, to compute homology in degrees $2, 1, 0$, let us analyze the tail end of the M-V sequence. Writing it from $\tilde{H}_2(K)$ to $\tilde{H}_0(S^1)$ we get

$$0 \to \tilde{H}_2(K) \to \tilde{H}_1(S^1) \to \tilde{H}_1(\{\text{pt}\}) \oplus \tilde{H}_1(S^1 \vee S^1) \to \tilde{H}_1(K) \longrightarrow \tilde{H}_0(S^1)$$

which is equivalent to

$$0 \to \tilde{H}_2(K) \xrightarrow{\phi} \mathbb{Z} \to 0 \oplus (\mathbb{Z} \oplus \mathbb{Z}) \to \tilde{H}_1(K) \longrightarrow 0$$

Recall what ϕ is from the proof of Theorem 10.1: it is applying the induced map on homology, coming from the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$, to the cycles in $A \cap B$ i.e. $\phi : H_1(A \cap B) \to H_1(A) \oplus H_1(B)$ where $\phi(\gamma) = (i_A(\gamma), -i_B(\gamma)) = (0, -(a+b+a-b)$. Recall that $A \cap B \cong S^1$. The first zero is clear. The second part follows from "pushing" the circle to the outer sides and tracing it.

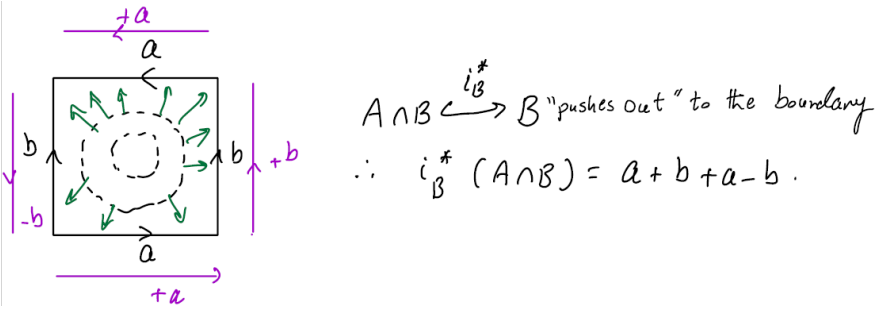


FIGURE 22. Computing the homology of the Klein bottle via M-V sequence

Therefore $\phi(1) = (0, (0, 2))$ i.e. ϕ is injective hence $\tilde{H}_2(K) \cong \ker(\phi) = 0$ which yields

$$0 \to \mathbb{Z} \xrightarrow{\phi} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\zeta} \tilde{H}_1(K) \longrightarrow 0 \text{ where } \text{Im}(\phi) = (0, 2)$$

The map ζ is surjective since we have a s.e.s therefore

$$\tilde{H}_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/(\text{Im}(\phi)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$$

Finally, one observes that $\tilde{H}_0(K) \cong 0$.

11. Excision and the Homology of a good pair

Monday March 9th:

"I want to make a hole in everything and penetrate it deep. I want to reach the heart of the earth. My love lies in there, a place where seedlings turn green and roots meet one another and creation continues even in disintegration. I think it has always been this way... in birth and then in death. I think my body is a temporary form. I want to reach its essence."

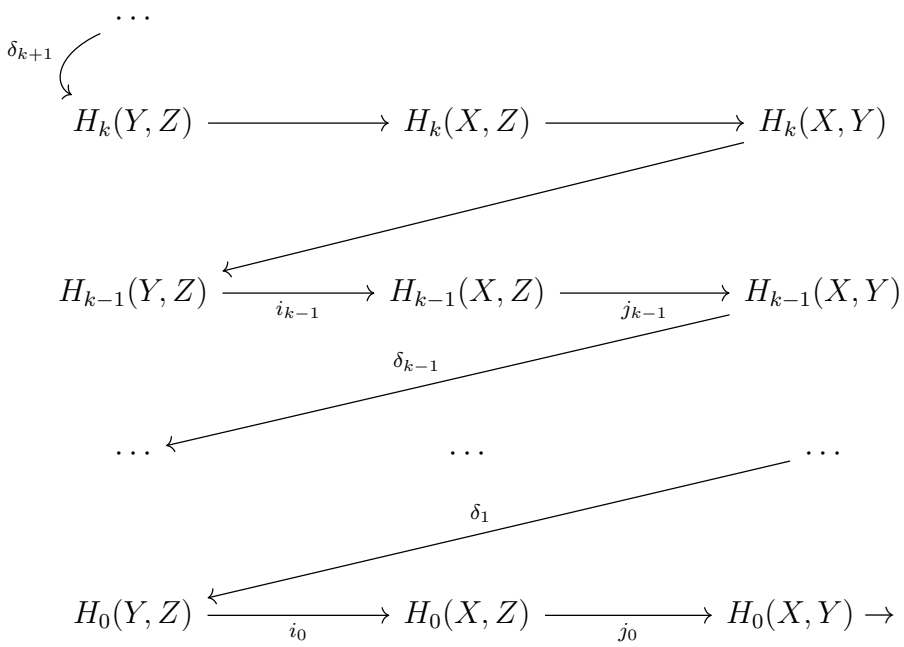
-Forough Farrokhzad.

The main goal of this section is to prove Theorem 9.1 which recall stated: if (X, A) is a good pair then $H_*(X, A) \cong \tilde{H}(X/A)$. Recall that if $Y \subset X$ then $C_k(X, Y) = \dfrac{C_k(X)}{C_k(Y)}$

Proposition 2 (LES for triples)**.** *Given a three topological spaces $Z \subset Y \subset X$ we have a s.e.s of chain complexes*

$$0 \longrightarrow C_*(Y, Z) \xrightarrow{i_*} C_*(X, Z) \xrightarrow{q_*} C_*(X, Y) \to 0$$

where i_* is given by the inclusion $i : Y \hookrightarrow X$ and q_* is well-defined since $C_*(Z) \subset C_*(Y)$. By the long exact sequence in homology theorem (aka snake lemma) we get the long exact sequence



One bit of notation: following the notation of Hatcher I am using $X - Z$ to denote $X \backslash Z$.

Theorem 11.1 (Excision Theorem)**.** *Suppose $Z \subset A \subset X$ such that $\overline{Z} \subset A$, that is the closure of Z is contained in the interior of A. Then the inclusion of the pairs $i : (X - Z, A - Z) \hookrightarrow (X, A)$ induces an isomorphism*

$$i_* : H_*(X - Z, A - Z) \longrightarrow H_*(X, A).$$

Proof. The crux of the proof comes from Proposition 1 which we never proved. Note that a drawing of our situation looks like:

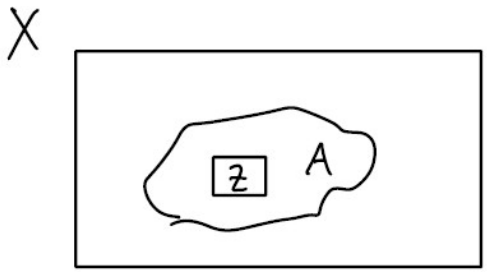


FIGURE 23. Sets involved in excision

First, I claim the theorem is equivalent to the following statement:

if $A, B \subset X$ such that $\text{Int}(A) \cup \text{Int}(B) = X$ then the inclusion $i : (B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism on relative homology.

To see statement implies theorem: take $Z = X - B$ equivalently $B = X - Z$. Then

$$(B, A \cap B) = (X - Z, A \cap (X - Z)) = (X - Z, A - Z). \tag{11.1}$$

Furthermore $\text{Int}(B) = X - \overline{Z}$. (If this is not clear to you look at the figure above.) Since the interiors of A and B cover X then $\overline{Z} \subset A$. To see the theorem implies the statement: take $B = X - Z$. By the previous argument it is clear that $\text{Int}(B) \cup \text{Int}(A) = X$ and Equation (11.1) holds verbatim.

With this in mind, we will prove the equivalent statement using the chain groups $C_*^{\mathcal{U}}(X)$; see Remark 11. I claim the following chain groups are isomorphic

$$C_*(B, A \cap B) \cong C_*^{\mathcal{U}}(X)/C_*(A) \tag{11.2}$$

where $\mathcal{U}$ corresponds to the open cover coming from the interiors of A and B. To see the aformentioned isomorphism we'll describe each. Fix $k \geq 0$. The chain group $C_k^{\mathcal{U}}(X)/C_k(A)$ are those k-simplicies which are contained in B but not entirely in A. Said differently, they are those k-simplicies contained in X but not completely in A since we are killing A. And since $\text{Int}(A) \cup \text{Int}(B) = X$, these are chains contained in B but not entirely in A. The left hand side $C_*(B, A \cap B) \cong C_k(B)/C_k(A \cap B)$ has the same description: chain contained in B but not entirely in A. The reasoning is similar, consider simplicies in B and kill those are that are $A \cap B$, then you are left with simplicies that are contained not entirely contained A; for otherwise they'd be in $A \cap B$ and therefore killed after moding out. Hence $C_*(B, A \cap B) \cong C_*^{\mathcal{U}}(X)/C_*(A)$.

By Proposition 1 we have $C_*^{\mathcal{U}}(X) \hookrightarrow C_*(X)$ induces $H_*^{\mathcal{U}}(X) \cong H_*(X)$ therefore

$$C_*^{\mathcal{U}}(X)/C_*(A) \hookrightarrow C_*(X)/C_*(A) \tag{11.3}$$

also induces an isomorphism on homology and we get $H_*^{\mathcal{U}}(X) \cong H_*(X, A)$. Plugging Equation (11.2) into Equation (11.3) we get

$$C_*(B, A \cap B) \cong C_*^{\mathcal{U}}(X)/C_*(A) \hookrightarrow C_*(X)/C_*(A)$$

which yields $H_*(B, A \cap B) \cong H_*^{\mathcal{U}}(X, A) \cong H_*(X, A)$ □

Wednesday March 11th:

"In order to grow...you must betray their expectations."

-Hayao Miyazaki.

We are now ready to give a fairly detailed proof of the following theorem.

Theorem 11.2. *If (X, A) is a good pair then $H_*(X, A) \cong \tilde{H}_*(X/A)$*

Proof. For every n we will construct a commutative diagram

$$\begin{array}{ccccc} H_n(X,A) & \xrightarrow{j_n} & H_n(X,V) & \xrightarrow{i_n} & H_n(X-A,V-A) \\ \downarrow{\scriptstyle q_n} & & \downarrow{\scriptstyle p_n} & & \downarrow{\scriptstyle r_n} \\ H_n(X/A,A/A) & \xrightarrow{j_n} & H_n(X/A,V/A) & \xleftarrow{i_n} & H_n(X/A-A/A,V/A-A/A) \end{array}$$

Claim: q_n is an isomorphism. Once established, we get that $H_n(X, A) \cong H_n(X/A, A/A) \cong \tilde{H}(X/A)$ where you will prove the final isomorphism in your homework.

To establish the claim, we show the outter maps are isomorphism. Let us inspect the maps in this diagram more closely.

Top map j_n: Considering the LES exact sequence in relative homology in degree n for the triplet $A \subset V \subset X$ we get

$$\cdots \quad \underbrace{H_n(V,A)}_{0} \longrightarrow H_n(X,A) \xrightarrow{j_n} H_n(X,V) \longrightarrow \underbrace{H_{n-1}(V,A)}_{0} \longrightarrow \cdots$$

To see the claimed groups $H_n(V, A)$ are indeed zero note that $v : (A, A) \hookrightarrow (V, A)$ is a homotopy equivalent of pairs therefore $H_*(A, A) \underset{v_*}{\cong} H_(V, A)$; however $H_*(A, A) = 0$ because the chain groups $C_n(A, A) = 0$ for all n. Therefore $j_n : H_n(X, A) \xrightarrow{\cong} H_n(X, V)$ is an isomorphism. An identical argument as above, applied to the tripl $A/A \subset V/A \subset X/A$, yields that the bottom j_n in the diagram is also an isomorphism i.e. $H_n(X/A, A/A) \cong H_n(X/A, V/A)$.

The top map i_n is induced from the map of pairs $i : (X - A, V - A) \longrightarrow (X, A)$. By Theorem 11.1, the induced map of i is an isomorphism. To see how this set up matches with the statement of Theorem 11.1, note that A here is Z from Theorem 11.1 and V here is A from Theorem 11.1; see Fig. 24.

An identical argument then show the bottom i_n is also an isomorphism. Finally, note that $r : (X - A, V - A) \longrightarrow (X/A - A/A, X/A - A/A)$ is a homeomorphism.This is a more

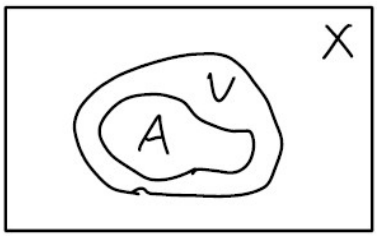


FIGURE 24. Sets involved in excision

general fact stemming from the assumption that A is closed; which we may assume from the definition of a good pair. (Proof is left as homework again.) See Figure 25.

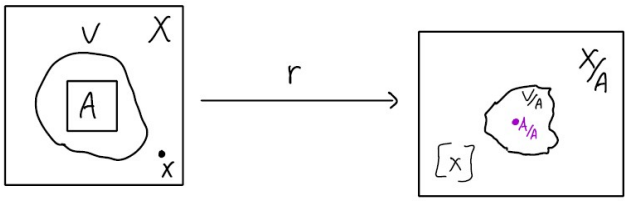


FIGURE 25. Projection map

Since homeomorphic spaces have the same homology groups we conclude that $r_n : H_n(X - A, V - A) \longrightarrow H_n(X/A - A/A, V/A - A/A)$ is also an isomorphism. Finally, we note that q_n are p_n are both induced from the quotient projection

$$q : (X, A) \longrightarrow (X/A, \underbrace{A/A}_{\text{pt in } X/A}) \qquad p : (X, V) \longrightarrow (X/A, V/A)$$

The commutativity of the diagram, which one can check, shows that both p_n and q_n are isomorphisms therefore

$$q_n = j_n^{-1} \circ i_n \circ r_n \circ i_n^{-1} \circ j_n.$$

□

We end this section by stating, but not proving, a fundamental result regarding homology groups.

Theorem 11.3. *For any topological space X we have*

$$H_*^{\Delta}(X) \cong H_*^{sing}(X).$$

We postpone the proof until later when we prove a more general theorem; see Theorem 13.1 and Corollary 7.

12. Degree of a map and the local degree formula

Friday March 13th:

"I'm aware, you know, that I and the people I love may perish in the morning. I know that. But there's light on our faces now."

-James Baldwin.

Throughout this section fix $n > 0$.

Definition 31. Given a map $f : S^n \longrightarrow S^n$, consider the induced homomorphism in dimension n

$$f_n : H_n(S^n) \cong \mathbb{Z} \longrightarrow H_n(S^n) \cong \mathbb{Z}$$
$$f_n([\tau]) = [f \circ \tau] \text{ where } \tau : \Delta^n \longrightarrow S^n.$$

If we let $\mathbb{1}$ be the generator of the source, then

$$f_n(\mathbb{1}) = d = d \cdot \underset{*}{\mathbb{1}}.$$

where $*$ is the generator of the target. The integer d is called the *degree of* $f : S^n \to S^n$.

Proposition 3 (Properties of the degree)**.** *Let* $f, g : S^n \longrightarrow S^n$.

(1) The degree is a homotopy invariant. That is, if $f \simeq g$ *then*

$$\deg(f) = \deg(g).$$

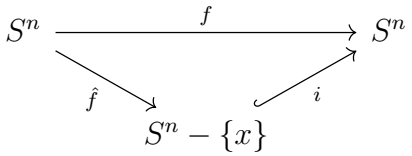
(2) The degree is multiplicative with respect to composition.

$$\deg(f \circ g) = \deg(f) \cdot \deg(g)$$

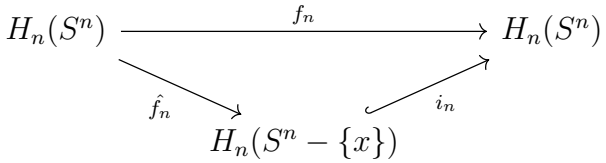
(3) The degree of the identity map is 1 *i.e.* $\deg(id_{S^n}) = 1$.

(4) If $f : S^n \longrightarrow S^n$ *is not surjective i.e.* $f(S^n) \subsetneq S^n$ *then* $\deg(f) = 0$.

Proof. The proof of 1 follows since the homotopic maps inuce the same map on homology. The proof of 2 is just the result of matrix multiplication and 3 is an observation. To see 4, let $x \in S^n - f(S^n)$, then f factors through to the inclusion $i : (S^n - \{x\}) \hookrightarrow S^n$ i.e.



and the induced map in homology yields



But since $S^n - \{x\} \cong \mathbb{R}^n$, which is contractible, the assumption that $n > 0$ implies f_n is the zero map. □

Example 33. Let us compute the degree of $A : S^n \longrightarrow S^n$ given by $A(\zeta) = -\zeta$. Here is a picture of this map in $\mathbb{R}^2$.

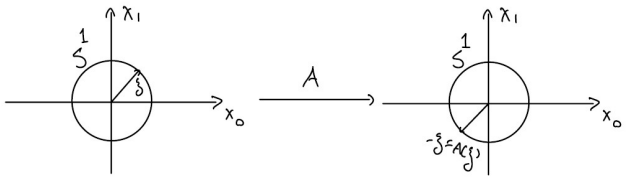


FIGURE 26. The map A in $\mathbb{R}^2$

Recall that $S^n \subset \mathbb{R}^{n+1}$ therefore we can think of A as $A(x_0, x_1, \ldots, x_n) = (-x_0, -x_1, \ldots, x_n)$. Let r_i denote the i-th reflection, that is $r_i(x_0, \ldots, x_i, \ldots, x_n) = (x_0, \ldots, -x_i, \ldots, x_n)$ and note that $A = \underbrace{r_0 \circ r_1 \circ \cdots \circ r_n}_{\text{n+1 terms}}$. We claim that $\deg(r_i) = -1$ which would imply $\deg(A) = (-1)^{n+1}$. I will prove the claim that $\deg(r_i) = -1$ for $n = 1$ and $i = 1$ and leave the general case as homework.

Consider the following delta complex structure on S^1

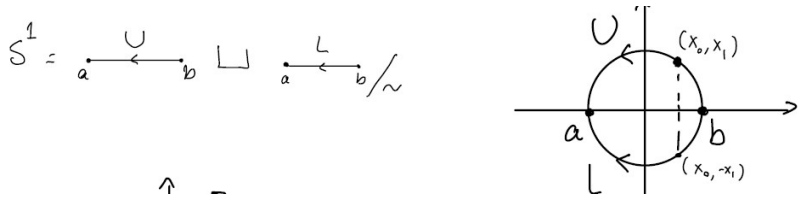


FIGURE 27. Δ-complex structure on S^1

Note that $r_1(x_0, x_1) = (x_0, -x_1)$ which is just reflection over the x-axis. Moreover, r_1 preserves the delta structure on S^1 as above since $r_1(U) = L$ and $r_1(L) = U$ i.e. r_1 maps simplicies to simplicies.

We already know that $H_1^\Delta(S^1) \cong \mathbb{Z}$. One can show that that the first homology group is actually generated by $U - L$ that is

$$H_1^\Delta(S^1) \cong \mathbb{Z}\langle U - L\rangle \xrightarrow{(r_1^\Delta)_*} \mathbb{Z}\langle U - L\rangle \cong H_1^\Delta(S^1)$$

Remember that the induced map on homology is simply applying the map to the class representative, in this case the generator.

$$(r_1^\Delta)_*(\underbrace{U - L}_{\mathbb{1}}) = r_1(U) - r_1(L) = L - U = -(\underbrace{U - L}_{\mathbb{1}}).$$

Therefore $\deg(r_1) = -1$. In your homework, you will generalize this to $n > 1$.

Let us give an application of Example 33. Recall that a map between topological spaces $f : X \longrightarrow X$ has a fixed point if and only if $f(x_0) = x_0$ for some $x_0 \in X$. The map A in the statement corresponds to A from Example 33.

Theorem 12.1. *If $f : S^n \longrightarrow S^n$ has no fixed points then $f \simeq A$, in particular* $\deg(f) = (-1)^{n+1}$

Proof. We show that the assumption $f(\zeta) \neq \zeta$ for all $\zeta \in S^n$ leads to a homotopy between f and A. Define $H(\zeta, t) = (1-t)f(\zeta) - t\zeta$ for any $0 \leq t \leq 1$ and $\zeta \in S^n$. Note that H is clearly continuous as it is linear. Claim: $H(\zeta, t) \neq 0$ for any t and ζ.

$$(1-t)f(\zeta) - t\zeta = 0 \iff (1-t)f(\zeta) = t\zeta \iff ||(1-t)f(\zeta)|| = ||t\zeta||$$
$$\iff |1-t|\underbrace{||f(\zeta)||}_{1} = |t|\underbrace{||\zeta||}_{1} \iff |1-t| = |t| \underset{0\leq t\leq 1}{\iff} t = \frac{1}{2}.$$

But $t = \frac{1}{2}$ implies $\frac{1}{2}f(\zeta) - \frac{1}{2}\zeta = 0$ i.e. $f(\zeta) = \zeta$ for some ζ which is a contradiction. Thus $H(\zeta, t) \neq 0$, therefore it makes sense to divide by $||H(\zeta, t)||$. Finally we define the following

$$\widehat{H} : S^n \times I \longrightarrow S^n \qquad \widehat{H}(\zeta, t) = \frac{H(\zeta, t)}{||H(\zeta, t)||}$$

Note that $\hat{H}$ is well-defined and continuous, moreover it defines a homotopy between f and A as

$$\widehat{H}(\zeta, 0) = \frac{f(\zeta)}{||f(\zeta)||} = f(\zeta)$$

$$\widehat{H}(\zeta, 1) = \frac{-\zeta}{||-\zeta||} = \zeta = A(\zeta).$$

Therefore $f \simeq A$ and the result follows. □

Monday March 16th:

"Maybe working on the little things as dutifully and honestly as we can is how we stay sane when the world is falling apart."

-Haruki Murakami.

We have now come to perhaps one of the most unfortunate-named theorems in mathematics: the hairy ball theorem (pretty hairy name if you ask me....).

Theorem 12.2 (Hairy Ball Theorem)**.** *Let $V : S^n \longrightarrow \mathbb{R}^{n+1}$ be a map such that the image of every point $V(\zeta)$ is tangent to S^n at the point ζ; that is for all $\zeta \in S^n$ $V(\zeta) \perp \zeta$. Then $V(\zeta) \neq 0$ for all $\zeta \in S^n$ if and only if n is odd.*

A function V that satisfies the hypothesis of theorem i.e. $V(\zeta) \perp \zeta$ for all $\zeta \in S^n$ is called *a vector field on S^n*. Note that the statement in the backwards directions says if n is odd, then there exists a function V with such property, it does NOT say that every vector field has this property.

Example 34. Here is a picture of the theorem in action when $n = 1$ i.e. $v(x, y) = v(x, -y)$

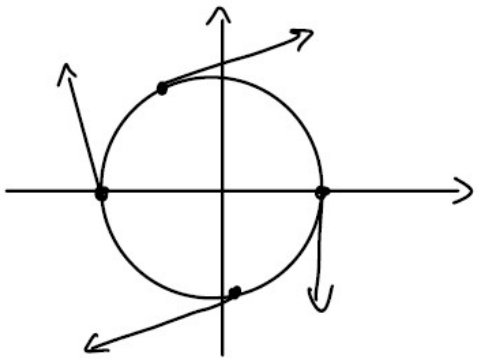


FIGURE 28. Hairy Ball Theorem for $n = 1$

Proof. ($\Rightarrow$) The assumption allows us to define the map

$$H : S^n \times [0,1] \longrightarrow S^n \qquad H(\zeta, t) = \cos(\pi t)\zeta + \sin(\pi t)\frac{V(\zeta)}{||V(\zeta)||}$$

Note that $H(\zeta, t)$ is well-defined i.e. ends up in S^n because

$$||H(\zeta,t)||^2 = \langle H(\zeta,t), H(\zeta,t)\rangle = \cos^2(2\pi t)\cdot||\zeta||^2 + \sin^2(2\pi t)\frac{||V(\zeta)||^2}{||V(\zeta)||^2} = 1.$$

Moreover H is a homotopy between id_{S^n} and A because

$$H(\zeta, 0) = \zeta = \mathrm{id}_{S^n} \quad H(\zeta, 1) = -\zeta = A(\zeta).$$

Therefore $\mathrm{id}_{S^n} \simeq A$ which implies $1 = \deg(\mathrm{id}_{S^n}) = (-1)^{n+1} = \deg(A)$ i.e. n is odd.
($\Leftarrow$) Suppose $n = 2k-1$ is odd. Define

$$V : S^n \longrightarrow S^n \qquad V(\zeta_1, \zeta_2, \ldots, \zeta_{2k}) = (-\zeta_2, \zeta_1, -\zeta_4, \zeta_3, \ldots, -\zeta_{2k}, \zeta_{2k-1})$$

and note that $\langle V(\zeta), \zeta\rangle = 0$ i.e. $V(\zeta) \perp \zeta$ for all $\zeta \in S^n$. Moreover $||V(\zeta)|| = ||\zeta|| = 1$ which implies $||V(\zeta)|| \neq 0$ for all $\zeta \in S^n$. □

In order to define the local degree we need a map with the following property. Suppose $f : S^n \longrightarrow S^n$ is a map with the property that there exists an $x_0 \in S^n$ with an open neighborhood U such that $f|_U$ is a homeomorphism onto $f(U) = V$. Note that $f(x_0) \in f(U)$ and U needs to be homeomorphic to an open neighberhood of $\mathbb{R}^n$. let $y_0 = f(x_0)$. Next, we show that

$$(f|_U)_* : H_*(U, U - \{x_0\}) \longrightarrow H_*(V, V - \{y_0\}) \tag{12.1}$$

is an isomorphism. By letting $X = S^n$, $A = S^n - \{y_0\}$ and $Z = S^n - U$ excision Theorem 11.1 applies and we get

$$H_*(U, U - \{x_0\}) \cong H_*(S^n, S^n - \{x_0\}) \cong H_*(V, V - \{x_0\}) \tag{12.2}$$

where the last isomorphism follows since the choice of open set U was arbitrary. Moreover, applying the long exact sequence in homology to the pair $(S^n, S^n - \{x_0\})$ we get that

$$\cdots \longrightarrow \underbrace{\tilde{H}_k(S^n - \{x_0\})}_{0} \longrightarrow \tilde{H}_k(S^n) \longrightarrow H_k(S^n, S^n - \{x_0\})$$

$$\underbrace{\tilde{H}_{k-1}(S^n - \{x_0\})}_{0} \longrightarrow \cdots$$

therefore

$$H_k(U, U - \{x_0\}) \cong H_*(S^n, S^n - \{x_0\}) = \begin{cases} \mathbb{Z} & k = n \\ 0 \text{ otherwise} \end{cases}$$

(Note this is not a typo, I actually mean homology and not reduced homology. You can convincen yourself that the last group is indeed 0.) Therefore Eq. (12.1) is

$$\begin{aligned} (f|_U)_* &: H_*(U, U - \{x_0\}) \longrightarrow H_*(V, V - \{y_0\}) \\ (f|_U) &: H_n(U, U - \{x_0\}) \cong \mathbb{Z} \longrightarrow \mathbb{Z} \cong H_n(V, V - \{x_0\}) \end{aligned} \tag{12.3}$$

Definition 32 (Local Degree)**.** The induced map on n-th homology of the pair $(U, U-\{x_0\})$ in Equation (12.3) is again given by sending $\mathbb{1}$ to $d \cdot \mathbf{1}$ for some integer d. The local degree of f at x_0 is defined as

$$\deg_{x_0}(f) = (f|_U)_n(\mathbb{1}).$$

Since $f|_U$ is a homeomorphism we have that $\deg_{x_0}(f) = \pm 1$.

Wednesday March 18th:

"Don't search for the answers, which could not be given to you now, because you would not be able to live them...the point is, to live everything. Live the questions now. Perhaps then, someday far in the future, you will gradually, without even noticing it, live your way into the answer."

-Rainer Maria Rilke (Austrian Poet).

It turns out that under mild assumptions we can compute the degree using the local degree.

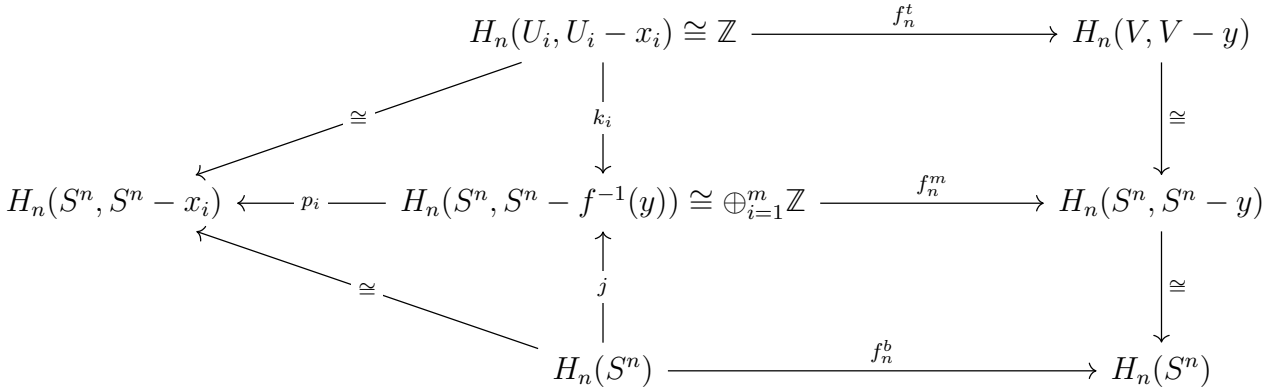
Theorem 12.3 (Local Degree Formula)**.** *Given a map $f : S^n \longrightarrow S^n$ such that for some $y \in S^n$ we have:*

(1) $f^{-1}(\{y\}) = \{x_1, \ldots, x_m\}$ is a finite set in S^n,
(2) and for all $1 \leq i \leq m$ there exists $U_i \subset S^n$, neighborhood of x_i such that $f|_{U_i}$ is a homeomorphism onto V_i where $y \in V_i$.

Then

$$\deg(f) = \sum_{i+1}^{m} \deg_{x_i}(f).$$

Proof. First, note that we can choose $U_i's$ to be disjoint by choosing smaller and smaller neighborhoods; a fact which will be crucial later. Define $V = \bigcap_{i=1}^{m} V_i$ and note that $U_i \hookrightarrow V$. For each *fixed index* i there exists a commutative diagram as follows, which we will analyze in depth.



□

First note that all three horizontal maps are induced by f since f induces maps on pairs as follows:

$$f : (U_i, U_i - x_i) \longrightarrow (V, V - y) \Rightarrow \text{ induces } f_n^t$$
$$f : (S^n, S^n - f^{-1}(y)) \longrightarrow (S^n, S^n - \{y\}) \Rightarrow \text{ induces } f_n^m$$

and the map f_n^b is explained in Definition 31. The superscript t,m and b is there to refer to their place in the diagram only.

The vertical map k_i is induced by the natural inclusion $(U_i, U_i - x_i) \hookrightarrow (S^n, S^n - f^{-1}(y))$; clearly this is an inclusion of pairs since $U_i - \{x_i\} \hookrightarrow S^n - f^{-1}(y)$. The commutativity of the diagram, which is the only detail I am skipping, in particular the commutativity of the top triangle implies p_i is projection onto the i-th factor.

The two isomorphisms in the upper half of the diagram are justified in Equation (12.2) and its preceding paragraph. Moreover, Equation (12.3) explains the groups being isomorphic to $\mathbb{Z}$. (Ultimately, this is by Excision.)

The two isomorphisms in the bottom half are justify simply by using the long exact sequence in homology, which we explained above.

Next, we will show $H_n(S^n, S^n - f^{-1}(y)) \cong \oplus_{i=1}^m \mathbb{Z}$. Let $U = \bigcup_{i=1}^m U_i$. By excision applied to the inclusion of pairs $(U, U - f^{-1}(y)) \hookrightarrow (S^n, S^n - f^{-1}(y))$, where in the statement of excision Theorem 11.1 we take $X = S^n$, $A = S^n - f^{-1}(y)$ and $Z = S^n - U$, it follows that

$$H_n(S^n, S^n - f^{-1}(y)) \cong H_n(U, U - f^{-1}(y))$$

It is not hard to verify the condition $\overline{Z} \subset \overset{\circ}{A}$ in the statement of Theorem 11.1 is satisfied; try drawing the situation. Next, we will show

$$H_n(U, U - f^{-1}(y)) \cong \bigoplus_{i=1}^m H_n(U_i, U_i - \{x_i\})$$

from the claimed isomorphism follows; we have shown above that $H_n(U_i, U_i - \{x_i\}) \cong \mathbb{Z}$. This exactly where U_i's being disjoing comes into play. This fact yields

$$\frac{C_(U)}{C_n(U - f^{-1}(y))} \cong \oplus \bigoplus_{i=1}^m \frac{C_n(U_i)}{C_n(U_i - \{x_i\})} \longrightarrow H_n(U, U - f^{-1}(y)) \cong \bigoplus_{i=1}^m H_n(U_i, U_i - \{x_i\}) \cong \bigoplus_{i=1}^m \mathbb{Z}$$

The only remaining map is j. Using the commutativity of the lower triangle we have $p_i(j(1)) = 1$ for all i therefore $j(1) = (1, 1, \ldots, 1)$. Moreover, note that $\sum_i^m k_i(1) = (1, 1, \ldots, 1)$. This sum is sensible since each k_i has its image in $\bigoplus_{i=1}^m \mathbb{Z}$. Thus $j(1) = \sum_i^m k_i(1) = (1, 1, \ldots, 1)$ Using the commutativity of the top square we have

$$f_n^m(k_i(1)) = f_n^t(1) = \deg f|_{x_i} \Rightarrow f_n^m(\sum_i^m k_i(1)) = \sum_{i=1}^m \deg(f|_{x_i}).$$

Finally using $j(1) = \sum_i^m k_i(1) = (1, 1, \ldots, 1)$ for the second equality, the commutativity of the bottom triangle for the third equality, and the definition of degree for the ultimate equality we get

$$\sum_{i=1}^m \deg(f|_{x_i}) = f_n^m(\sum_i^m k_i(1)) = f_n^m(j(1)) = f_n^B(1) = \deg(f)$$

Monday March 30th:

"I can only connect deeply or not at all."

-Anais Nin.

13. Cell Complexes

Definition 33. A cell complex is a topological space $\bigcup_{n=0}^{m} X^n$ where X^n is a closed subspace called an *n-skeleton of X* satisfying:

(1) X^0 is a finite set of points;
(2) X^n is constructed inductively:

$$X^n = \left(X^{n-1} \sqcup_{\varphi_{\alpha_1}} D^n_{\alpha_1}\right) \sqcup_{\varphi_{\alpha_2}} D^n_{\alpha_2} \cdots \sqcup_{\varphi_{\alpha_k}} D^n_{\alpha_k}$$

where $\varphi_{\alpha_i} : \partial D^n_{\alpha_i} \to X^{n-1}$, $D^n_{\alpha_i} \cong D^n \subset \mathbb{R}^n$ where D^n is the unit disc in $\mathbb{R}^n$ and each map φ_{α_i} can be extended to

$$\phi^n_{\alpha_i} : D^n_{\alpha_i} \to X^n$$
$$\phi^n_{\alpha_i}|_{\partial D^n_{\alpha_i}} = \varphi_{\alpha_i}.$$

Recall that the disjoint union glued along the map is defined as

$$Z \sqcup_\varphi D^n = \frac{Z \sqcup D^n}{\varphi(x) \sim \underbrace{x}_{\in \partial D^n_{\alpha_i}}} \text{ where } \varphi : \partial D^m \to Z.$$

- Each $e^n_{\alpha_i} = \phi^n_{\alpha_i}(D^n_{\alpha_i})$ is called an n-cell of X.
- The map $\phi^n_{\alpha_i}$ is the characteristic map of $e^n_{\alpha_i}$.
- The map $\phi^n_{\alpha_i}$ is the attaching map of $e^n_{\alpha_i}$.
- If $X = X^m$, we say X is finite dimensional and $\dim X = m$.

As a matter of terminology, we note that CW-complexes and Cell-complexes are the same.

(3) * For any i and n, an n-cell $e^n_{\alpha_i} \cong \text{int}(D^n)$ i.e. an n-cell is homeomorphic to the interior of an n-disk.

A short note on item 3*: The * is to denote that this condition may be redundant. That an n-cell is homeomorphic to the *interior* of an n-cell is well-known (see [Mun84] for example.) and I strongly suspect that it is a consequence of the first two items, however I was not able to show 3 is a consequence of 1 and 2, thus im adding it to the definition. It either makes redundant or correct, either of which is better than incorrect!

Example 35. Every Δ-complex is also a cell/CW complex.

Example 36. [Cell structure on S^1]

$$X^0 = \{x_0\}, \quad X^1 = X^0 \sqcup_\varphi \underbrace{[0,1]}_{D^1} = \frac{\{x_0\} \sqcup [0,1]}{\varphi(x) \sim x}$$

where $\varphi : \underset{\{0,1\}}{\partial D^1} \to \underset{\{x_0\}}{X^0}, \phi : \underset{t}{D^1} \underset{\to}{\to} \underset{(\cos(2\pi t), \sin(2\pi t))}{X^1}$.

Cell structure of $S^1 = \{e^0, e^1\}$.

Example 37. [Cell structure(s) on S^n] First cell structure on S^n

$$X^0 = \{x_0\} = X^1 = \cdots = X^{n-1}$$
$$X^n = \frac{\{x_0\} \sqcup D^n}{\varphi(x) \sim x}, \quad \varphi : \partial D^n \to \{x_0\}$$
$$\text{Cell structure of } S^n = \{e^0, e^n\}.$$

Second cell structure on S^n

$$X^0 = \{x_0\} = X^1 = \cdots = X^{n-2}$$
$$X^{n-1} = \{x_0\} \sqcup_\varphi D^{n-1} = \frac{\{x_0\} \sqcup D^{n-1}}{\varphi(x) \sim x} \text{ where } \varphi : \partial D^{n-1} \to \{x_0\},$$
$$X^n = \left(X^{n-1} \sqcup_{\psi_1} D_1^n\right) \sqcup_{\psi_2} D_2^n \text{ where } \psi_i : \underbrace{\partial D_i^n}_{\cong S^{n-1}} \xrightarrow[id]{} \underbrace{X^{n-1}}_{\cong S^{n-1}}$$
$$\text{Cell structure of } S^n = \{e^0, e^{n-1}, e_1^n, e_2^n\}.$$

Wednesday April 1st and Friday April 3rd:

"To make living itself an art, that is the goal."

-Henry MIller.

Notice what follows are for two days of lecture combined.

Example 38. [Cell structure on genus g surface] Let Σ_g denote the genus g surface i.e. g donuts connected to each other, a surface with g donut-like holes.

$g = 1$:

$$\Sigma_1^0 \cong T^2 \cong S^1 \times S^1, \quad \Sigma_1^0 = \{x\}$$
$$\Sigma_1^1 = \{v\} \sqcup_{\varphi_a} a \sqcup_{\varphi_b} b, \quad \varphi_a, \varphi_b : \partial D^1 \to \{v\}$$
$$\Sigma_1^2 = \Sigma_1^1 \sqcup_\psi D^2, \quad \psi : \underbrace{\partial D^2}_{S^1} \to \Sigma_1^1 \text{ where } \psi = a * b * \overline{a} * \overline{b}$$
$$\text{Cell structure on } \Sigma_1 = \{e^0, e_0^1, e_1^1, e^2\}.$$

$g = 2$:

$$\Sigma_2^0 = \{v\}$$
$$\Sigma_2^1 = \{v\} \sqcup_{\varphi_a} a \sqcup_{\varphi_b} b \sqcup_{\varphi_c} c \sqcup_{\varphi_d} d, \quad \varphi_i : \partial D^1 \to \{v\}, \ \ i \in \{a, b, c, d\},$$
$$\Sigma_1^2 = \Sigma_2^1 \sqcup_\psi D^2, \quad \psi : \underbrace{\partial D^2}_{S^1} \to \Sigma_2^1 \text{ where } \psi = [a, b] * [c, d], \ \ [a, b] = a * b * \overline{a} * \overline{b},$$
$$\text{Cell structure on } \Sigma_2 = \{e^0, e_0^1, e_1^1, e_2^1, e_3^1, e^2\}.$$

For $g \geq 1$:

$$\Sigma_g^0 = \{v\}$$
$$\Sigma_g^1 = \{v\} \sqcup_{\varphi_{a_1}} a_1 \ \sqcup_{\varphi_{b_1}} b_1 \cdots \sqcup_{\varphi_{a_i}} a_i \ \sqcup_{\varphi_{b_i}} b_i, \quad \varphi_{a_i}, \varphi_{b_i} : \partial D^1 \to \{v\}, \ \ 1 \leq i \leq g,$$
$$\Sigma_g^2 = \Sigma_g^1 \sqcup_\psi D^2, \quad \psi : \underbrace{\partial D^2}_{S^1} \to \Sigma_g^1 \text{ where } \psi = [a_1, b+1] * \cdots * [a_g, b_g]$$

Cell structure on $\Sigma_g = \{e^0, e_0^1, e_1^1, e_2^1, e_3^1, \ldots, e_{2g-2}^1, e_{2g-1}^1, e^2\}$.

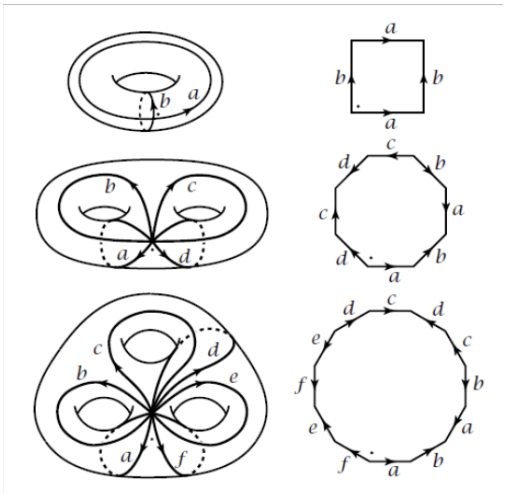


FIGURE 29. Surfaces of genus $1, 2, 3$

Lemma 13. *If X is a finite CW-complex, then*

(1) (X^n, X^{n-1}) is a good pair.

(2) $H_k^{sing} \cong \bigoplus_{i=1}^n H_k(D_{\alpha_i}^n, \partial D_{\alpha_i}^n) \cong \begin{cases} \mathbb{Z}\langle \phi_{\alpha_1}^n, \ldots, \phi_{\alpha_n}^n \rangle & \text{if } k = n \\ 0 & \text{if } k \neq n \end{cases}$

(3) The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism

$$i_* : H_k(X^n) \to H_k(X) \text{ for } k < n,$$

and is surjective when $k = n$.

Proof. The proof of part (1) is left as homework.

2) By part a) we have $H_k(X, X^{n-1}) \cong \widetilde{H}_k(X/X^{n-1})$ where

$$\frac{X^n}{X^{n-1}} \cong \frac{\bigsqcup_i D_{\alpha_i}^n}{\bigsqcup_i \partial D_{\alpha_i}^n} = \left(\frac{D_{\alpha_1}^n}{\partial D_{\alpha_1}^n}\right) \vee \cdots \vee \left(\frac{D_{\alpha_r}^n}{\partial D_{\alpha_r}^n}\right)$$

I explained the first isomorphism in gory details in class via two different justifications. First, this isomorphism follows because removing the $n-1$-skeleton is akin to removing the boundaries of the D_{α_i}'s since they are attached to the X^{n-1}-skeleton. This is the left hand side quotient in plain english i.e.

$$\frac{X^n}{X^{n-1}} = \frac{\left(\frac{X^{n-1} \sqcup D_{\alpha_1}^n \sqcup \cdots \sqcup D_{\alpha_k}^n}{\varphi_{\alpha_i}(x) \sim x \text{ for all } i}\right)}{X^{n-1}} \cong \frac{\bigsqcup_i D_{\alpha_i}^n}{\bigsqcup_i \partial D_{\alpha_i}^n}$$

The second explanation is that this isomorphism is induced from $(\twoheadrightarrow \circ \Phi)$ where

$$\Phi = \phi^n_{\alpha_1} \sqcup \ldots \sqcup \phi^n_{\alpha_r} : D^n_{\alpha_1} \sqcup \ldots \sqcup D^n_{\alpha_1} \to X^n$$
$$\twoheadrightarrow \circ \Phi : \phi^n_{\alpha_1} \sqcup \ldots \sqcup \phi^n_{\alpha_r} \to \frac{X^n}{X^{n-1}}$$

And one can show that $\ker(\twoheadrightarrow \circ \Phi) = \bigsqcup_i \partial D^n_{\alpha_i}$. Recall that

$$\widetilde{H}_k \left(\bigvee_{i=1}^{r} \frac{D^n_{\alpha_i}}{\partial D^n_{\alpha_i}} \right) \cong \bigoplus_{i=1}^{r} H_k^{\text{sing}}(D^n_{\alpha_i}, \partial D^n_{\alpha_i})$$

Finally, note that one can identify D^n with Δ^n, as an example the filled triangle (Δ^2) and D^2 are homeomorphic thus $(D^n_{\alpha_i}, \partial D^n_{\alpha_i}) \cong (\Delta^n, \partial \Delta^n) \cong \mathbb{Z}$. Furthermore, the generators of $H_k^{\text{sing}}(D^n_{\alpha_i}, \partial D^n_{\alpha_i})$ are the characteristic maps and item (2) follows. Parts c and d are repeated applications of the long exact sequence in homology for a pair.
3) Writing the LES in homology for (X^n, X^{n-1}) yields

$$\cdots \to H_{k+1}(X^n, X^{n-1}) \longrightarrow H_k(X^{n-1}) \xrightarrow{i_*} H_k(X^n) \longrightarrow H_k(X^n, X^{n-1}) \to \cdots$$

if $k \neq n$ or $k \neq n-1$ both groups $H_{k+1}(X^n, X^{n-1}) = H_k(X^n, X^{n-1}) = 0$ (by part a) and we get that i_* is an isomorphism i.e. $H_k(X^{n-1}) \cong H_k(X^n)$. In particular whenever $k > n$ we can repeat the above LES and get

$$H_k(X^n) \cong H_k(X^{n-1}) \cong H_k(X^{n-2}) \cong \cdots H_k(X^n) \cong 0.$$

4) Now if we write the long exact sequence in Homology for (X^{n+1}, X^n) we get

$$\cdots \to H_{k+1}(X^{n+1}, X^n) \longrightarrow H_k(X^n) \xrightarrow{i_*} H_k(X^{n+1}) \longrightarrow H_k(X^{n+1}, X^n) \to \cdots$$

If we assume $k < n \iff k+1 < n+1$, part a again implies that $H_{k+1}(X^{n+1}, X^n) = H_k(X^{n+1}, X^n) = 0$ so we get

$$H_k(X^n) \cong H_k(X^{n+1}) \cong \cdots \cong H_k(X) \text{ where } X = X^m \text{ and } m = \dim(X).$$

(Credit to Izzy to izzy for pointing out the missing detail here) To actually see what the previous chain of isomorphisms follows, note that much like the first part we can repeat (look further into) the LES but the key point is $n = k < m$ where $\dim(X) = m$ and thus we go up to $X^m = X$ only. At the $k = n$ index we have that the right most group is 0 by part a i.e. $H_k(X^{n+1}, X^n) = 0$ so we get that

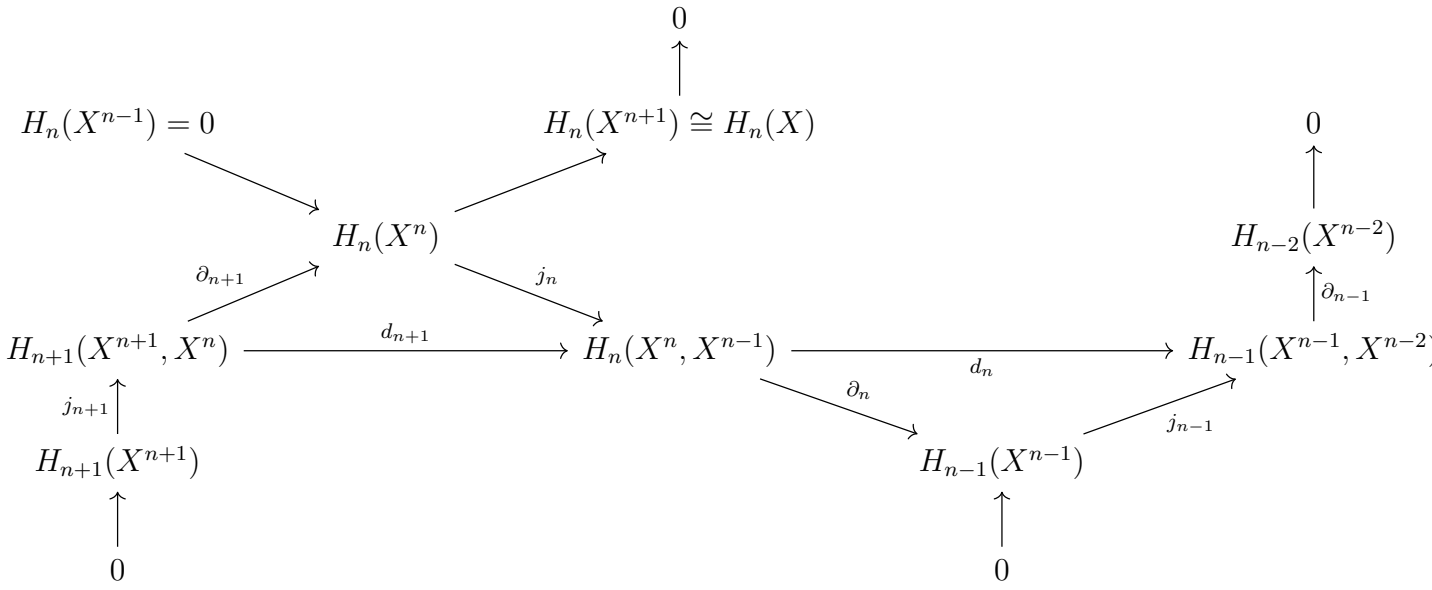
$$H_k(X^n) \xrightarrow{i_*} H_k(X^{n+1}) \longrightarrow 0 \Rightarrow H_k(X^n) \xrightarrow{i_*} H_k(X^{n+1}) \cong H_k(X)$$

where the last isomorphism follows from the chain of isomorphisms above. □

Definition 34. For $X = \bigcup_n X^n$ a cell complex we define the n-th CW chain groups as

$$C_n^{\text{cw}} = H_k^{\text{sing}}(X^n, X^{n-1}) \cong \mathbb{Z}\langle e^n_{\alpha_1}, \ldots, e^n_{\alpha_r} \rangle.$$

We define the chain maps $d_* := j_{*-1} \circ \partial_*$ where j_* and ∂_* are coming from the long exact sequence in of pairs as follows (assuming $* = n$ and $* = n+1$):



For spacing reasons I have to draw this a little weird, the class diagram is better. Since $\operatorname{Im}(j_n) \subseteq \ker(\delta_n)$ we get

$$d_n \circ d_{n+1} = (j_{n-1} \circ \partial_n) \circ (j_n \circ \partial_{n+1}) = j_{n-1} \circ (\underbrace{\partial_n \circ j_n}_{0}) \circ \partial_{n+1} = 0.$$

Thus $(C_*^{\mathrm{CW}}(X), d_*)$ is a complex.

$$H_*^{\mathrm{CW}} = H_*(C_*^{\mathrm{CW}}(X), d_*) = \frac{\ker(d_*)}{\operatorname{Im}(d_{*+1})} \tag{13.1}$$

Some observations about the CW homology

(1) $H_n(X) = 0$ if X is a CW complex with n-cells.
(2) More generally, if X is a CW complex with k n-cells, then $H_n(X)$ is generated by at most k elements. This is because $H_n(X^n, X^{n-1})$ is a free abelian group on k generators, thus the subgroup $\ker(d_n)$ and subsequently the quotient $\ker(d_n)/\operatorname{Im}(d_{n+1})$ is generated by at most k generators.
(3) If X is a CW complex having no two of its cells in adjacent dimensions, then $H_n(X)$ is free abelian with basis in one-to-one correspondence with the n-cells of X. This is because the cellular boundary maps d_n are automatically zero in this case.

Monday April 6th: (Did review/summary of everything up to this point.)

"I have divested of meaning."

-Joan Halifax.

Wednesday April 8th:

"The world around us is in a sea change, and I think the glory of art is that it cannot only survive change, it can lead it."

-Robert Redford.

Let us demonstrate two examples of item (3) of the observation above.

Example 39. Let $X \cong S^n$ with $n > 1$. We know a cell structure of S^n is given by $\{e^0, e^n\}$ and the sequence is given by

$$0 \to \mathbb{Z}\langle e^n \rangle \to 0 \to \cdots \to 0 \to \mathbb{Z}\langle e^0 \rangle \to 0$$

and item (3) above implies

$$H_*^{CW}(S^n) = \begin{cases} \mathbb{Z} & \text{if } * = n, 0 \\ 0 & \text{else} \end{cases}$$

Example 40. Let $X = \mathbb{CP}^n \cong \frac{\mathbb{C}^{n+1}}{\overline{Z} \sim \lambda \overline{Z}}$ where $\lambda \neq 0$ and $\lambda \in \mathbb{C}$. This space admits a cell structure of the form $\{e^0, e^2, e^4, \ldots, e^{2n}\}$ i.e. only even-dimensional single cells. We will justify this fact later. Thus its CW chain complex looks like

$$0 \to \mathbb{Z}\langle e^{2n} \rangle \to 0 \to \mathbb{Z}\langle e^{2n-2} \rangle \to \ldots \to \mathbb{Z}\langle e^2 \rangle \to 0 \to \mathbb{Z}\langle e^0 \rangle \to 0$$

and therefore by item (3) its homology groups look like

$$H_*^{\text{CW}} = \begin{cases} \mathbb{Z} & \text{if } * = 0, 2, 4, \ldots, 2n \\ 0 & \text{else} \end{cases}$$

Theorem 13.1. $H_n^{CW}(X) \cong H_n^{sing}(X)$.

Proof. Using the diagram above we see that

$$\frac{H_n^{\text{sing}}(X^n)}{\text{Im}(\partial_{n+1})} \cong H_n^{\text{sing}}(X^{n+1}) \cong H_n^{\text{sing}}(X). \tag{13.2}$$

Since j_n is injective, it maps $\text{Im}(\partial_{n+1})$ isomorphically onto its image, that is

$$\text{Im}(\partial_{n+1}) \cong j_n(\text{Im}(\partial_{n+1})) \cong \text{Im}(j_n \circ \partial_{n+1}) = \text{Im}(d_{n+1})$$

where the last equality follows by the definition of d_{n+1}. Next note that

$$H_n^{\text{sing}}(X^n) = \frac{H_n^{\text{sing}}(X^n)}{0} = \frac{H_n^{\text{sing}}(X^n)}{\ker(j_n)} \cong j_n(H_n^{\text{sing}}(X^n)) = \text{Im}(j_n) = \ker(\partial_n) \overset{*}{=} \ker(d_n)$$

where the equality $*$ follows since j_{n-1} is injective i.e.

$$x \in \ker(d_n) \iff d_n(x) = 0 \iff (j_{n-1} \circ \partial_n)(x) = 0 \iff \partial_n(x) = 0 \iff x \in \ker(\partial_n)$$

Plugging the above two equations into the left hand side of Equation (13.2) we get

$$\frac{H_n^{\text{sing}}(X^n)}{\text{Im}(\partial_{n+1})} \cong H_n^{\text{sing}}(X^{n+1}) \cong H_n^{\text{sing}}(X)$$
$$\frac{H_n^{\text{sing}}(X^n)}{\text{Im}(\partial_{n+1})} \cong \frac{\ker(d_n)}{\text{Im}(\delta_{n+1})} = H_n^{\text{CW}}(X)$$

□

Corollary 7. *Let X be any topological space and $n \geq 0$.*

$$H_n^{simp}(X) \cong H_n^{\Delta}(X) \cong H_n^{CW}(X) \cong H_n^{sing}(X)$$

Proof. This follows because Simplicial and delta complexes are both examples of CW complexes. □

Next, we will try to come up with a recipe for determining the matrices of the of the maps

$$d_n : C_n^{\mathrm{CW}}(X) \cong \mathbb{Z}\langle e_\alpha^n \rangle \to \mathbb{Z}\langle e_\beta^{n-1} \rangle \cong C_{n-1}^{\mathrm{CW}}(X)$$

When $n = 0$, there is nothing to do as $d_0 = 0$. When $n = 1$ we have the following

$$\begin{array}{ccc} H_1(X^1, X^0) & \longrightarrow & H_1(X^0, \emptyset) \cong H_0(X^0) \\ \downarrow \cong & & \downarrow \cong \\ d_1 : \bigoplus_\alpha H_1(D_\alpha^1, \partial D_\alpha^1) & \longrightarrow & \mathbb{Z}\langle e_1^0, \ldots, e_m^0 \rangle \end{array}$$

Let's apply $d_1([\phi_\alpha^1]) = j_1(\partial_1(\phi_\alpha^1)) = \phi_\alpha^1|_{\{1\}} - \phi_\alpha^1|_{\{0\}}$
The last equality is not obvious but if we workout the maps in the long exact sequence for homology, we see that j_1 turns out to be the identity map and ∂_1 turns out to be the same as the simplicial boundary map.
Friday April 10th:

"A certain type of perfection can only be realized through a limitless accumulation of the imperfect."

-Haruki Murakami.

Recall the main difficulty in dealing with CW complexes is determining the maps. The recipe for d_n when $n \geq 2$ requires some machinery we developed previously, namely the degree of a map $f : S^n \longrightarrow S^n$.

Lemma 14 (Matrix of d_n for $n > 1$)**.** *Let X be a finite CW complex. The n-th chain complex map d_n is given by*

$$d_n : \mathbb{Z}\langle e_\alpha^n \mid \alpha = 1, \ldots, k \rangle \longrightarrow \mathbb{Z}\langle e_\beta^{n-1} \mid \beta = 1, \ldots, m \rangle$$
$$d_n(e_\alpha^n) = \sum_{\beta=1}^{m} d_{\alpha\beta} \epsilon_\beta^{n-1}$$

where $d_{\alpha\beta}$ is defined as the degree of the map $g_\beta \circ \varphi_\alpha^n$ with φ_α^n the attaching map of D_α^n and g_β defined as collapsing the complement of e_β^{n-1} to a point.

$$\underbrace{\partial D_\alpha^n}_{S^{n-1}} \xrightarrow{\varphi_\alpha^n} X^{n-1} \xrightarrow{g_\beta} S_\beta^{n-1}$$

Therefore the matrix of d_n is given by

$$[d_n]_{m \times k} = [d_{\alpha\beta}] \text{ where } 1 \leq \alpha \leq k \quad 1 \leq \beta \leq m$$

Remark 12. Given a continuously differentiable map $f : S^1 \longrightarrow S^1$, Let $y_0 \in S^1$. Using the local degree formula Definition 32 we get

$$\deg(f) = \sum_{x_i \in f^{-1}(y_0)} \deg_{x_i} f = \sum_{x_i \in f^{-1}(y_0)} \operatorname{sign}(f'(x_i))$$

where the last equality follows because

$$\deg_{x_i}(f) = \operatorname{sign}(f'(x_i)).$$

Note the previous equality is well-defined as y_0 is a regular value of f that is $f'(x_i) \neq 0$ for all $x_i \in f^{-1}\{y_0\}$. (But I am not fully justifying it.)

Example 41. Let $X = \{v_0\} \sqcup_\varphi e^1 \sqcup_\psi e^2$ with φ a continuous function. Then $\deg(\psi) = k$ =number of times ∂e^2 wraps around e^1. Thus the CW complex of X looks like

$$0 \to \mathbb{Z}\underset{e^2}{\langle e^2\rangle} \xrightarrow[\longrightarrow]{\psi} \mathbb{Z}\underset{k\cdot e^1}{\langle e^1\rangle} \xrightarrow{0} \mathbb{Z}\langle v_0\rangle \to 0 \Longrightarrow H_i^{\mathrm{CW}}(X) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}/k\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{if } i \geq 2 \end{cases}$$

Example 42. Let $X = ((\underbrace{S^1}_{a} \vee \underbrace{S^1}_{b}) \sqcup_{\phi_1} e_1^2) \sqcup_{\phi_2} e_2^2$ where $S^1 \vee S^1$ is joined at v_0, $\phi^1 = a^5 * b^{-3}$ and $\phi_2 = b^3 * (a * b)^{-2}$.

$$0 \to \mathbb{Z}\langle e^2\rangle \xrightarrow{\psi} \mathbb{Z}\underset{k\cdot e^1}{\langle e^1\rangle} \xrightarrow{0} \mathbb{Z}\langle v_0\rangle \to 0$$

Using Remark 12 one can comute the following:

$$\deg(g_a \circ \phi_1) = 5, \quad \deg(g_b \circ \phi_1) = -3, \quad \deg(g_a \circ \phi_2) = -2, \quad \deg(g_b \circ \phi_2) = 1$$

Therefore

$$[d_2] = \begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 \\ -3 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Longrightarrow \operatorname{Im}(d_2) \cong \mathbb{Z}^2 \ ; \ \ker(d_2) \cong \{0\}$$

$$H_k^{\mathrm{CW}}(X) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & \text{else} \end{cases} \Longrightarrow \tilde{H}_k(X) = 0 \text{ for all } k \Rightarrow \tilde{H}_*(X) = 0$$

A space whose reduced homologies are all zero is called an "acyclic" space.

Monday April 13th:

''One should not only photograph things for what they are but for what else they are."
-Milnor White.

Example 43. Recall the definition of the n dimensional real projective space

$$\mathbb{R}P^n := \frac{\mathbb{R}^{n+1} - \{0\}}{v \sim \lambda v, \ \lambda \neq 0} = \frac{S^n}{v \sim -v \ ; ||v|| = 1}$$

For $n = 0$ we have $\mathbb{R}P^0 = \dfrac{\mathbb{R} - \{0\}}{v \sim \lambda v} = \{\mathrm{pt}\}$. Equivalently, $\mathbb{R}P^0 = \dfrac{S^0 = \{-1, 1\}}{v \sim -v} = \{\mathrm{pt}\}$
Here is a picture

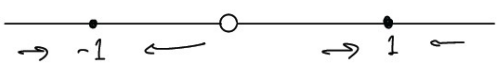


FIGURE 30. $\mathbb{R}P^0$

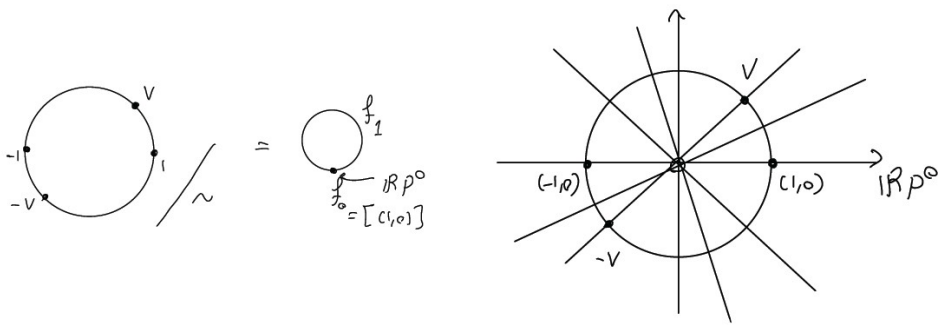


FIGURE 31. $\mathbb{R}P^1$

For $n = 1$ we have $\mathbb{R}P^1 = \dfrac{\mathbb{R}^2 - \{0\}}{V \sim \lambda v}$. Alternatively, we can see this from $\dfrac{S^1}{v \sim -v}$ which yields the exact same picture as in Figure 31.

The upshot is that we can observe that

$$\mathbb{R}P^1 = \underbrace{\mathbb{R}P^0}_{\{\text{pt}\}} \sqcup_\phi [-1, 1]$$

where ϕ is a continuous map from $\partial f_1 \longrightarrow f_0$. For $n = 2$ we get

$$\mathbb{R}P^2 = \mathbb{R}P^1 \sqcup_{\phi_2} f^2$$

where $f^2 \cong D^2$ and can be identified with the upper hemisphere. The map $\phi_2 = \text{id} \circ A$ is given as the following composition:

$$S^1 \cong \partial f^1 \hookrightarrow \mathbb{R}^2 - \{0\} \xrightarrow{A} \frac{\mathbb{R}^2 - \{0\}}{v \sim \lambda v} = \frac{S^1}{v \sim -v} = \mathbb{R}P^1$$

Here, identity corresponds to the inclusion map $\hookrightarrow$. Here is a picture of this composition and $\mathbb{R}P^2$:

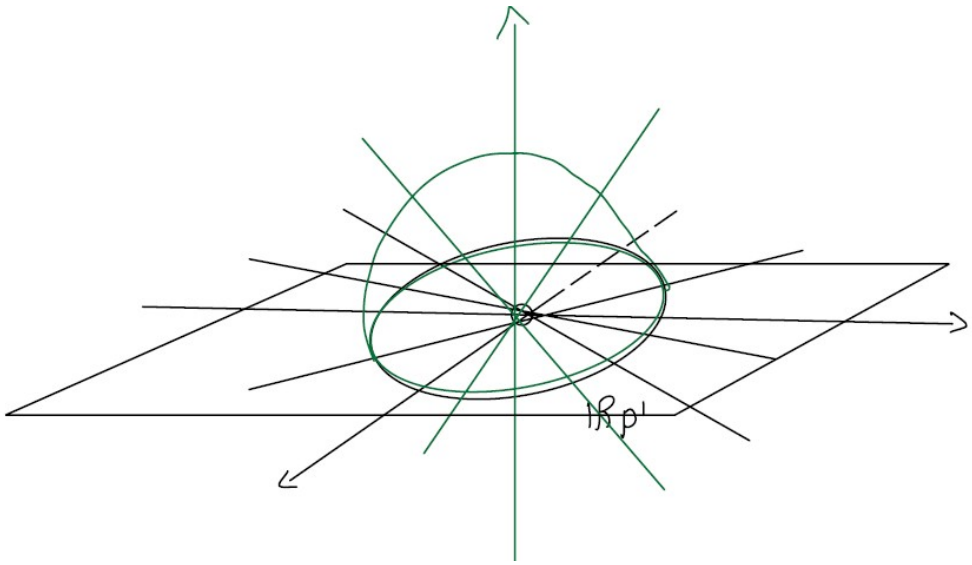


FIGURE 32. $\mathbb{R}P^2$

Note that $\deg(\phi_2) = \deg(\mathrm{id}) + \deg(A) = 1 + (-1)^2 = 2$. A small note here: technically we defined the degree of a map $f : S^n \to S^n$ but the above composition goes from S^1 to $\frac{S^1}{v \sim -v}$. Harsh raised this issue in class. The definition of degree here is the same, you consider the induced map on $f_* : H_1(S^1) \to H_1(\frac{S^1}{v \sim -v})$ and that gives you the degree. (reference?) In fact the above picture is exactly computing this degree. We can induct on this procedure and observe that

$$\mathbb{R}P^n \cong \mathbb{R}P^{n-1} \sqcup_{\phi_n} f_n, \quad f_n \cong D^n, \quad \phi_n := \partial f_n \hookrightarrow \mathbb{R}^n - \{0\} \xrightarrow{A} \mathbb{R}P^n \text{ i.e } \phi_n = A \circ \hookrightarrow$$

$$\deg(\phi_n) = \deg(\mathrm{id}) + \deg(A) = 1 + (-1)^n = \begin{cases} 0 & \text{if n-odd} \\ 2 & \text{if n-even} \end{cases}$$

Recall that the degree of the map allowed us to determine the maps in the CW complex directly (see Lemma 14). The chain complex then is as follows

$$0 \longrightarrow \mathbb{Z}\langle f_n \rangle \xrightarrow{\phi_n} \mathbb{Z}\langle f_{n-1} \rangle \xrightarrow{\phi_{n-1}} \mathbb{Z}\langle f_{n-2} \rangle \xrightarrow{\phi_{n-2}} \cdots \xrightarrow{\phi_3=0} \mathbb{Z}\langle f_2 \rangle \xrightarrow{\phi_2=\cdot 2} \mathbb{Z}\langle f_1 \rangle \xrightarrow{\phi_1=0} \mathbb{Z}\langle f_0 \rangle \xrightarrow{\phi_0=0} 0$$

The only difference between the odd and even case case is homology in degree n. If n is even we get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/(2\mathbb{Z}) = \mathbb{Z}_2 & \text{if } k < n \text{ and k is odd} \\ 0 & \text{else} \end{cases}$$

and if n is odd we get

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z}/(2\mathbb{Z}) = \mathbb{Z}_2 & \text{if } k < n \text{ and k is odd} \\ \mathbb{Z} & \text{if } k = n \\ 0 & \text{else} \end{cases}$$

We apply a similar construction to $\mathbb{C}P^n$.

Example 44. Recall the definition of the n dimensional complex projective space

$$\mathbb{C}P^n := \frac{\mathbb{R}^{n+1} - \{0\}}{z \sim \lambda z,\ \lambda \neq 0} \cong \frac{S^{2n+1}}{z \sim \lambda z\ ; |z| = 1\ ; \lambda = e^{i\theta}}$$

Our goal is to build up $\mathbb{C}P^n$ like $\mathbb{R}P^n$ by inductively attaching disks to a smaller dimensional $\mathbb{C}P^n$. We will show $\mathbb{C}P^n \cong \mathbb{C}P^{n-1} \sqcup_{\varphi_n} D^{2n}$ where φ_n will be defined as follows: Consider

$$D^{2n} \xrightarrow{i} (w_0, w_1, \ldots, w_{n-1}, \sqrt{q - |w|^2}) \in S^{2n+1} \twoheadrightarrow S^{2n+1}/z \sim \lambda z = \mathbb{C}P^n$$

Here is a picture of this map:

Let $\phi_n := (\twoheadrightarrow \circ i)$. Then we define the gluing map φ_n as follows

$$\varphi_n : \partial D^{2n} \longrightarrow \mathbb{C}P^{n-1}$$

$$\partial D^{2n} = \{w \mid |w| = 1\} \cong S^{2n-1} \xrightarrow{\phi_n|_{\partial D^{2n}}} (w_0, \ldots, w_{n-1}, 0) \in S^{2n-1} \xrightarrow{q} \frac{S^{2n-1}}{\sim} = \mathbb{C}P^{n-1}$$

So to be precise, $\varphi_n = q \circ (\phi_n|_{\partial D^2 n})$. One small note: in Definition 33 we technically defined the gluing map φ_n (going from ∂D^{2n} to X^{n-1}) first and then the characteristic map ϕ_n as

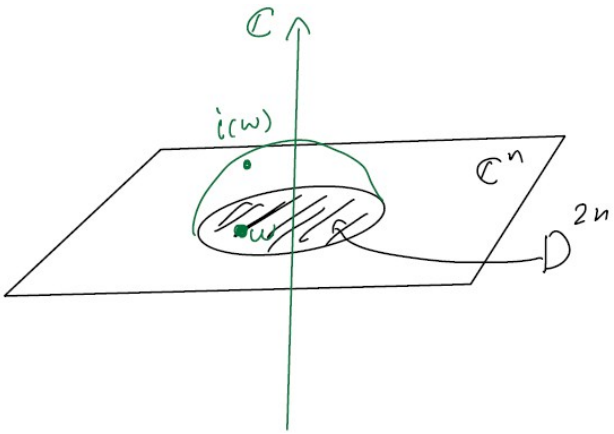


FIGURE 33. $\mathbb{C}P^n$

an extension of the gluing map (going from D^{2n} to X^n) in such a way that $\phi_n|_{\partial D^{2n}} = \varphi_n$. Ultimately, you can do this in either way so long as it's done in a continuous fashion. Moreover, you can always define the characteristic map first, look at the restriction and then work backwards. The world is your oyster so long as you remain continuous.

The upshot is we have now shown

$$\mathbb{C}P^n \cong \mathbb{C}P^{n-1} \sqcup_{\varphi_n} D^{2n} \cong (\mathbb{C}P^0 \sqcup_{\varphi_1} D^2) \sqcup_{\varphi_2} D^4 \sqcup_{\varphi_3} \cdots \sqcup_{\varphi_n} D^{2n} \Rightarrow \mathbb{C}P^n = \{e^0, e^2, e^4, \ldots, e^{2n}\}$$

$$0 \to \mathbb{Z}\langle e^{2n}\rangle \xrightarrow{0} \mathbb{Z}\langle e^{2n-2}\rangle \xrightarrow{0} \cdots \mathbb{Z}\langle e^2\rangle \xrightarrow{0} \mathbb{Z}\langle e^0\rangle \xrightarrow{0} 0$$

$$H_k^{\mathrm{CW}} = \begin{cases} \mathbb{Z} & \text{if } 0 \leq k \leq 2n \ k - \text{even} \\ 0 & \text{else} \end{cases}$$

Wednesday April 15th:

"Just slow things down and it becomes more beautiful."

-David Lynch.

14. Homology with coefficients

For transparency: The first few definitions and examples of this section are taken from UNL's Algebra-901 taught in Fall 2019. Throughout this section we take R to be a commutative ring with unity mainly because I want to develop the background slightly more general. For all practical purposes you can take $R = \mathbb{Z}$. Moreover, throughout we assume M and N are R modules (not necessarily finitely generated.) There is a fairly straightforward way to generalize all of this to non-commutative rings, we then have to consider left and right modules.

Definition 35 (R-biadditive function)**.** For R modules M and N, and A an abelian group, a function

$$b : M \times N \longrightarrow A \tag{14.1}$$

is called R-biadditive if the following conditions hold:

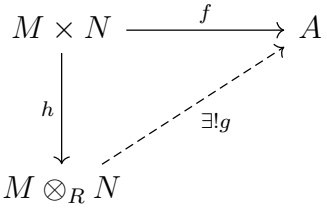
(1) $b(m + m', n) = b(m, n) + b(m', n)$ for all $m, m' \in M$ and $n \in N$,
(2) $b(m, n + n') = b(m, n) + b(m, n')$ for all $m \in M$, $n, n' in N$, and

(3) $b(mr, n) = b(m, rn) = rb(m, n)$ for all $m \in M$, $n \in N$ and $r \in R$.

Example 45. Some examples include of bilinear maps are:

- $f : R \times R \to R$, $f(r, s) = rs$ i.e. the multiplication map
- for an R-module M, $f : R^2 \times M \longrightarrow M$, $f((r, s), m) = (rm, sm)$
- For an ideal $I \subset R$ and a module M, $f : (R/I) \times M \longrightarrow M/IM$ via $f(\overline{r}, m) = \overline{rm}$.

Definition 36 (Tensor Product as a universal object)**.** Let R, M and N be as above. An abelian group $M \otimes_R N$ together with an R-biadditive map $h : M \times N \to M \otimes_R N$ is called the tensor product of M and N if it has the following universal property: for any abelian group A and R-biadditive map $f : M \times N \to A$, there exists a unique group homomorphism $g : M \otimes_R N \to A$ such that $f = g \circ h$ i.e. the following diagram commutes for a unique g



Note the tensor product is unique up to isomorphism.

Definition 37 (Theorem/Definition: Tensor product, generators and relations)**.** With the set up above, $M \otimes_R N$ is the free group generated by all expressions of the form $m \otimes n$ for $m \in M$ and $n \in N$ subject to the following relations:

(1) $(m + m') \otimes n = m \otimes n + m' \otimes n$ for all $m, m' \in M$ and $n \in N$,
(2) $m \otimes (n + n') = m \otimes n + m \otimes n'$ for all $m, m' \in M$ and $n, n' \in N$, and
(3) $(mr) \otimes n = m \otimes (rn)$ for all $m \in M$, $n \in N$ and $r \in R$.

More succinctly,

$$M \otimes_R N = \frac{\bigoplus_{(m,n) \in M \times N} \mathbb{Z} \cdot (m \otimes n)}{(Y)}$$

where

$$Y = \{(m+m') \otimes n - m \otimes n - m' \otimes n\} \cup \{m \otimes (n+n') - m \otimes n - m \otimes n'\} \cup \{(mr) \otimes n - m \otimes (rn)\}$$

Further we define $h : M \times N \longrightarrow M \otimes_R N$ to be the function $h(m, n) = m \otimes n$. Then the pair $(M \otimes_R N, h)$ defined as above the tensor product of M and N.

The elements of the form $m \otimes n$ are called simple tensors and the tensors product is all $\mathbb{Z}$-linear combinations of simple tensors module the aforementioned relations.

Proposition 4 (Some properties of the tensor product)**.** *Let M, N and Q be R-modules.*

(1) $M \otimes_R N \cong N \otimes_R M$
(2) $(N_1 \oplus \cdots \oplus N_r) \otimes M \cong \oplus_i N_i \otimes M$
(3) $M \otimes (N \otimes Q) = (M \otimes N) \otimes Q$
(4) $R \otimes_R N \cong N$ *via* $(r \otimes n) \to rn$
(5) $R/I \otimes N \cong N/I \cdot N$ *via* $\overline{r} \otimes n \to \overline{rn}$
(6) For a pair of homomorphisms $f : A \longrightarrow A'$ *and* $g : B \longrightarrow B'$ *we have*

$$f \otimes g : M \otimes N \longrightarrow M' \otimes N' \text{ via } f \otimes g(m \otimes n) = f(m) \otimes f(n)$$

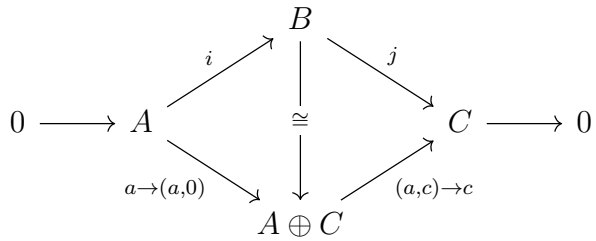
We are now switching the name of our modules to A, B and C but they are still R modules over R. Recall that a short exact sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

is split if and only if $B \cong A \oplus C$.

Lemma 15 (The splitting Lemma)**.** *The following statements are equivalent:*

(1) There exists an isomorphism $B \cong A \oplus C$ making the following diagram commute



i.e. the s.e.s splits.

(2) There exists a homomorphism $p : B \to A$ such that $p \circ i = id_A$

$$0 \to A \underset{p}{\overset{i}{\rightleftarrows}} B \xrightarrow{j} C \to 0$$

(3) There exists a homomorphism $s : C \to B$ (called a section) such that $j \circ s = id_C$

$$0 \to A \xrightarrow{i} B \underset{s}{\overset{j}{\rightleftarrows}} C \to 0$$

The proof is straightforward so I omit it.

Let us state an application of split short exact sequences to good pairs. Given a good pair (X, A) and a retraction $r : X \to A$ such that $i \circ r = \mathrm{id}_A$ the long exact sequence in homology yields

$$\cdots \xrightarrow{\delta_{n+1}} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\delta_n} \cdots$$

Note that r induces a map on homology as well and we have $r_* : H_n(X) \to H_n(A)$, Moreover,

$$r_* \circ i_* = (r \circ i)_* = (\mathrm{id}_A)_* = \mathrm{id}_{H_n(A)}$$

therefore $\ker(i_*) = 0$ i.e. the image of the previous map, namely δ_{n+1} is also zero and this is true for any arbitrary n. Thus for all n we get the short exact sequence

$$0 \to H_n(A) \underset{r_*}{\overset{i_*}{\rightleftarrows}} H_n(X) \xrightarrow{j_*} H_n(X, A) \to 0$$

which is split by Lemma 15 therefore

$$H_n(X) \cong H_n(A) \oplus H_n(X, A) \cong H_n(A) \oplus \tilde{H}_n(X/A).$$

Homology with coefficients Going forward we set $R = \mathbb{Z}$. Fix an abelian group G. In Definition 21 we defined singular chains (with $\mathbb{Z}$ coefficients) on X as:

$$C_k(X) := \bigoplus_{\sigma:\Delta^k \to X} \mathbb{Z} = \{\Sigma_i k_i \sigma_i \mid k_i \in \mathbb{Z}\}$$

To generalize this, we replace $\mathbb{Z}$ with G and set

$$C_k(X;G) := \bigoplus_{\sigma:\Delta^k\to X} G = \{\Sigma_i g_i\sigma_i \mid g_i \in G\}$$

and for a pair (X, A) we set

$$C_k(X, A; G) := \frac{C_k(X;G)}{C_k(A;G)}$$

One can expand the definition of ∂_k, which is defined as before via restriction to faces, to show that

$$\partial_{k-1} \circ \partial_k(g \cdot \sigma) = g(\partial_{k-1} \circ \partial_k(\sigma)) = 0$$

we get that $(C_*(X, A; G); \partial_*)$ is a chain complex and

$$H_k(X, A; G) := \frac{\ker \partial_k}{\operatorname{Im} \partial_{k+1}}$$

is called the homology of (X, A) with G-coefficients. Let us state two results without proof.
Friday April 17th:

"Chance encounters are what keep us going."

-Haruki Murakami.

Lemma 16.

$$C_k(X, A; G) \cong C_k(X, A) \otimes G$$

where $C_k(X, A) \otimes G = \{\Sigma_i g_i \otimes \sigma_i \mid g_i \in G\}$

It turns out that we can compute homology with coefficients in any group G using the homology groups we computed over $\mathbb{Z}$. This uses Tor groups which stands for Torsion groups. I will state the results first and then define these groups.

Theorem 14.1 (Universal Coefficient Theorem for homology (UCT)-Algebraic Version)**.** *If (C_*, ∂_*) is a chain complex of free abelian abelian groups, then for each n there is a (natural) short exact sequence of the form*

$$0 \to H_n(C_*) \otimes G \longrightarrow H_n(C_*; G) \longrightarrow \operatorname{Tor}(H_{n-1}(C_*), G) \to 0$$

which splits.

Theorem 14.2 (Universal Coefficient Theorem for homology (UCT)-Topological Version)**.** *If (C_*, ∂_*) where $C_* = C_*^{sing}(X, A)$, then for each n there is a (natural) short exact sequence of the form*

$$0 \to H_n(X, A) \otimes G \longrightarrow H_n(X, A; G) \longrightarrow \operatorname{Tor}(H_{n-1}(X, A), G) \to 0$$

which splits implying

$$H_n(X, A; G) \cong (H_n(X, A) \otimes G) \oplus \operatorname{Tor}(H_{n-1}(X, A), G)$$

We need some machinery and results to define the abelian group $\operatorname{Tor}(A, B)$. Let A and B be abelian groups.

Lemma 17. *If the sequenece of abelian groups*

$$A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

is exact then so is its tensor product with $- \otimes G$ *i.e. the following is exact*

$$A \otimes G \xrightarrow{i\otimes \mathbb{1}} B \otimes G \xrightarrow{j\otimes \mathbb{1}} C \otimes G \to 0$$

Remark 13. Note that the previous lemma does **not** extend to a split short exact sequence of the form

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

Here is an example to demonstrate this point. If we tensor the split s.e.s

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2 = i_2} \mathbb{Z} \to \mathbb{Z}_2 \to 0$$

with $\mathbb{Z}_2$ we get

$$0 \longrightarrow \underbrace{\mathbb{Z} \otimes \mathbb{Z}_2}_{\cong \mathbb{Z}_2} \xrightarrow{i_2 \otimes \mathbb{1}} \underbrace{\mathbb{Z} \otimes \mathbb{Z}_2}_{\cong \mathbb{Z}_2} \to \underbrace{\mathbb{Z}_2 \otimes \mathbb{Z}_2}_{\cong (\mathbb{Z}_2)^2} \to 0$$

Note this implies $\ker(i_2 \otimes \mathbb{1}) \cong \mathbb{Z}_2$ thus the s.e.s is not exact after tensoring with $\mathbb{Z}_2$.

Remark 14 (2-step free resolution of A). For any abelian group A there exists a s.e.s of the form

$$0 \to F_1 \xrightarrow{i} F_0 \xrightarrow{j} A \to 0$$

where $F_0 = \bigoplus_A \mathbb{Z}$, equivalently we can take the index of F_0 to be a generating set of A. The map j is defined as a free-extension of the map $a \to a : A \hookrightarrow A$. Define $F_1 = \ker(j)$ and that F_1 is also free since it is a subgroup of a free group therefore we have, by definition, a short exact sequence. This is called the 2-stage or 2-step free resolution of A.

Let B be any other abelian group. Applying $- \otimes B$ to the s.e.s in Remark 14 yields

$$F_1 \otimes B \xrightarrow{i\otimes \mathbb{1}} F_0 \otimes B \xrightarrow{j\otimes \mathbb{1}} A \otimes B \to 0 \tag{14.2}$$

Definition 38. The group $\mathrm{Tor}(A, B)$ is defined as $\ker(i \otimes \mathbb{1})$ in Equation (14.2).

Of course one needs to check that the previous definition is independent of the choice of the free resolution in Equation (14.2) but that turns out to be true.

Example 46. Based on the discussion above $\mathrm{Tor}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$

Proposition 5 (Some properties of $\mathrm{Tor}(A, B)$.)**.** *(1)* $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$.

(2) $\mathrm{Tor}(\oplus_i A_i, B) \cong \oplus_i \mathrm{Tor}(A, B)$

(3) $\mathrm{Tor}(A, B) = 0$ *if* A *or* B *is free, or more generally torsion free.*

(4) $\mathrm{Tor}(A, B) = \mathrm{Tor}(T(A), B)$ *where* $T(A)$ *is the torsion subgroup of* A.

(5) $\mathrm{Tor}(\mathbb{Z}_n, A) \cong \ker(A \xrightarrow{\cdot n} A)$.

(6) For each short exact sequence $0 \to B \to C \to D \to 0$ *there is a naturally associated exact sequence*

$$0 \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(A, C) \to \mathrm{Tor}(A, D) \to A \otimes B \to A \otimes C \to A \otimes D \to 0$$

Corollary 8. $H_n(X; k) \cong H_n(X) \otimes k$ *where* $k = \mathbb{R}, \mathbb{Q}, \mathbb{C}$.

Monday April 20th:

"Reading is solitude. One reads alone, even in another's presence."

-Italo Calvino.

Example 47. Let X be the Klein bottle. Recall that

$$H_n(X) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases} \Rightarrow H_n(X;\mathbb{R}) = \begin{cases} \mathbb{R} & \text{if } n = 0 \\ \mathbb{R} & \text{if } n = 1 \\ 0 & \text{if } n > 1 \end{cases}$$

To see the implication, note that $\mathbb{Z} \otimes \mathbb{R} \cong \mathbb{R}$ as $\mathbb{R}$ has the structure of a $\mathbb{Z}$-module. Using Proposition 5 we get

$$(\mathbb{Z} \oplus \mathbb{Z}_2) \otimes \mathbb{R} = (\mathbb{Z} \otimes \mathbb{R}) \oplus \underbrace{(\mathbb{Z}_2 \otimes \mathbb{R})}_{0} \cong \mathbb{R}.$$

Note that $\mathbb{Z}_2 \otimes \mathbb{R} = \ker(\mathbb{R} \xrightarrow{\times 2} \mathbb{R}) = 0$.

Lemma 18. $\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{\gcd(m,n)}$

Proof. It is clear that $1 \otimes 1$ generates $\mathbb{Z}_m \otimes \mathbb{Z}_n$ since $x \otimes y = xy(1 \otimes 1)$. Let $d = \gcd(m,n)$ and $d = mp + nq$ for some integers $p, q \in \mathbb{Z}$.

$$m(1 \otimes m) = n(1 \otimes 1) = 0 \Rightarrow d(1 \otimes 1) = mp(1 \otimes 1) + nq(1 \otimes 1) = 0$$

This implies $|Z_m \otimes \mathbb{Z}_n| \leq d$. To see the order is at least d, note that one can show the map

$$f : \mathbb{Z}_m \otimes \mathbb{Z}_n \longrightarrow \mathbb{Z}_d$$
$$(x \mod m) \otimes (y \mod n) \longrightarrow x \cdot y \mod d$$

is an onto homomorphism thus $|Z_m \otimes \mathbb{Z}_n| \geq d$. □

Corollary 9. *Let k be either $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. If $H_n(X)$ is finitely generated with betti number β_n then*

$$H_n(X) \cong \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{\beta_n} \oplus (\mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}})$$
$$H_n(X;k) \cong \underbrace{k \oplus \cdots \oplus k}_{\beta_n} \cong k^{\beta_n}$$
$$rank(H_n(X)) = \dim(H_n(X;k))$$

Corollary 10. *Let p be prime and assume $H_n(X)$ and $H_{n-1}(X)$ are finitely generated. Then $H_n(X;\mathbb{Z}_p)$ is a direct sum of $\mathbb{Z}_p$'s with the following contributions:*

(1) Each factor of $\mathbb{Z}$ in $H_n(X)$ gives a factor of $\mathbb{Z}_p$ in $H_n(X;\mathbb{Z}_p)$ ($\mathbb{Z} \otimes \mathbb{Z}_p \cong \mathbb{Z}_p$)
(2) Each copy of $\mathbb{Z}_{p^k}$ in $H_n(X)$, with $k \geq 1$, gives a $\mathbb{Z}_p$ factor in $H_n(X;\mathbb{Z}_p)$ ($\mathbb{Z}_p \otimes \mathbb{Z}_{p^k} \cong \mathbb{Z}_{\gcd(p,p^k)}$)
(3) Each copy of $\mathbb{Z}_{p^k}$ in $H_{n-1}(X)$ gives a $\mathbb{Z}_p$ factor in $H_n(X;\mathbb{Z}_p)$

Proof. To see (3) above, recall that

$$H_k(X;G) \cong (H_k(X) \otimes G) \oplus \mathrm{Tor}(H_{k-1}(X), G) \cong (H_k(X) \otimes G) \oplus \mathrm{Tor}(\mathrm{Tor}(H_{k-1}(X)), G).$$

□

Example 48.

$$H_k(\mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & \text{if } k=0 \\ \mathbb{Z}_2 & \text{if } k=1 \\ 0 & \text{if } k \geq 2 \end{cases} \longrightarrow H_k(\mathbb{R}P^2;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k=0 \\ \mathbb{Z}_2 & \text{if } k=1 \\ \mathbb{Z}_2 & \text{if } k=2 \\ 0 & \text{if } k=0 \end{cases}$$

where the $\mathbb{Z}_2$ in degree 2 is coming from $\mathrm{Tor}(H_1(\mathbb{R}P^2),\mathbb{Z}_2)$.

15. Künneth Formula

The main goal of this section is to compute the homology of a product space using the homology of each individual space. That is, how do we connect $H_*(X\times Y)$ to $H_*(X)$ and $H_*(Y)$.

Theorem 15.1. *There exists a $\mathbb{Z}$-bilinear map:*

$$H_i(X)\times H_j(Y) \longrightarrow H_{i+j}(X\times Y).$$

While I won't carefully prove this theorem let us contemplate how the map is constructed. It is induced by the following chain map:

$$\begin{aligned} C_i(X)\times C_j(Y) &\longrightarrow C_{i+j}(X\times Y) \\ (\sigma,\tau) &\longrightarrow \text{ Chain Representing } \sigma\times\tau \\ \sigma:\Delta^i\to X\ ;\ \tau:\Delta^j\to Y \longrightarrow \sigma\times\tau : \Delta^i\times\Delta^j &\longrightarrow X\times Y \\ \sigma\times\tau(t,u) &\longrightarrow (\sigma(t),\tau(u)) \end{aligned}$$

Using Definition 36 we then get a homomorphism on the tensor product

$$C_i(X)\otimes C_j(Y) \longrightarrow C_{i+j}(X\times Y)$$

and therefore a chain homomorphism between the chains

$$\begin{aligned} \big(C_*(X)\otimes C_*(Y),\delta_*^{\otimes}\big) &\xrightarrow{\Psi} \big(C_*(X\times Y),\delta_*^{X\times Y}\big) \\ (C_*(X)\otimes C_*(Y))_n &= \bigoplus_{i+j=n} C_i(X)\otimes C_j(Y) \text{ where } a\otimes b\in C_i(X)\otimes C_j(Y) \\ \delta_*^{\otimes}(a\otimes b) &:= (\delta_*^X a)\otimes b + (-1)^i a\otimes(\delta_*^Y b) \end{aligned} \tag{15.1}$$

One can check that $\delta_n^{\otimes}\circ\delta_{n+1}^{\otimes}=0$.

Wednesday April 22th:

"You have to invent life."

-Agnes Varda.

Theorem 15.2. *The chain homomorphism Ψ induces the isomorphism*

$$H_*(C_*(X)\otimes C_*(Y),\delta_*^{\otimes}) \cong H_*(X\times Y)$$

Thus to answer the question of interest, stated at the beginning of the section, it suffices to express $H_*(C_*(X)\otimes C_*(Y),\delta_*^{\otimes})$ in terms of $H_*(X)=H_*(C_*(X),\delta_*^X)$ and $H_*(Y)=H_*(C_*(Y),\delta_*^Y)$.

Remark 15. Note that the construction of the tensor product of two complexes is a more general construction. For two chain complexes (C_*, ∂_*) and (C'_*, ∂'_*) we can always form the chain complex $(C_* \otimes C'_*, \delta^{\otimes}_*)$ basically via the same recipe as before, that is

$$(C_* \otimes C'_*)_m = \bigoplus_{i+j=n} C_i \otimes C'_j$$
$$\partial^{\otimes}_*(\underset{\in C_i}{a} \otimes \underset{\in C'_j}{b}) = (\delta_i(a)) \otimes b + (-1)^i a \otimes \delta'_j b$$

Theorem 15.3 (Algebraic Künnet Formula)**.** *Suppose C_* and C'_* are complexes of free $\mathbb{Z}$-modules, then for each n there is a natural split short exact sequence of the form*

$$0 \to \bigoplus_{i=0}^{n} (H_i(C_*) \otimes H_{n-i}(C'_*)) \to H_n(C_* \otimes C'_*) \to \bigoplus_{i=0}^{n-1} \operatorname{Tor}(H_i(C), H_{n-i-1}(C')) \to 0$$

If in the statement of Theorem 15.3 we let $C_* = (C_*(X), \partial^X_*)$ and $C'_* = (C_*(Y), \partial^Y_*)$ we obtain the "topological Künneth Formula"

$$0 \to \bigoplus_{i=0}^{n} (H_i(X) \otimes H_{n-i}(Y)) \to H_n(X \times Y) \to \bigoplus_{i=0}^{n-1} \operatorname{Tor}(H_i(X), H_{n-i-1}(Y)) \to 0 \tag{15.2}$$

Note to get this equation, I am simultaneously replacing C_* and C'_* as well as invoking Theorem 15.2 to replace the middle term. Moreover, if we can show that the homology groups of X and Y, that is $H_*(X)$ and $H_*(Y)$, are torsion free, then the last term in Equation (15.2) goes away and we are left with the aesthetically pleasing formula

$$\bigoplus_{i=0}^{n} (H_i(X) \otimes H_{n-i}(Y)) \cong H_n(X \times Y) \tag{15.3}$$

Example 49. Let $X = Y = S^1$ and recall that $H_0(S^1) = H_1(S^1) = \mathbb{Z}$ and is zero otherwise. In particular, the homology groups are torsion free which puts us in the setting of Equation (15.3).

$$H_0(S^1 \times S^1) = H_0(S^1) \otimes H_0(S^1) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \text{(expected as this space is path connected)}$$
$$H_1(S^1 \times S^1) = (H_0(S^1) \otimes H_1(S^1)) \oplus (H_1(S^1) \otimes H_0(S^1)) = (\mathbb{Z} \otimes_{\mathbb{Z}}) \oplus (\mathbb{Z} \otimes_{\mathbb{Z}}) \cong \mathbb{Z} \oplus \mathbb{Z}$$
$$H_2(S^1 \times S^1) = (H_0(S^1) \otimes \underbrace{H_2(S^1)}_{0}) \oplus (H_1(S^1) \otimes H_1(S^1)) \oplus (\underbrace{H_2(S^1)}_{0} \otimes H_0(S^1)) = 0 \oplus \mathbb{Z} \oplus 0 = \mathbb{Z}$$

And for higher homology groups, we just end up getting zeros that is $H_k(S^1 \otimes S^1) = 0$ for all $k \geq 3$.

Example 50. Let $X = \mathbb{C}P^n \times \mathbb{C}P^m$. In general there are three approaches to consider in computing the homology groups $H_*(\mathbb{C}P^n \times \mathbb{C}P^m)$, the first of which we will use in this example.

(1) Build a "product cell" structure on $\mathbb{C}P^n \times \mathbb{C}P^m$.
(2) Use the isomorphism in Theorem 15.2
(3) Use the Künneth formula

Recall that $\mathbb{C}P^n$ was built with even degree cells and we showed

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & \text{if } k \leq 2n \text{ and } k\text{-even} \\ 0 & \text{else} \end{cases}$$

Once again, the homology groups are Torsion free and we are in the set up of Equation (15.2) if we wanted to use the Künneth formula directly. Now let $X = \mathbb{C}P^4 \times \mathbb{C}P^2$. We will use the cell structure on the individual spaces $\mathbb{C}P^4$ and $\mathbb{C}P^2$ as well as the recipe in Equation (15.1) to get

$C_*(\mathbb{C}P^4)$:

$$0 \to \overset{8}{\mathbb{Z}} \to 0 \to \overset{6}{\mathbb{Z}} \to 0 \to \overset{4}{\mathbb{Z}} \to 0 \to \overset{2}{\mathbb{Z}} \to 0 \to \overset{0}{\mathbb{Z}} \to 0$$

$C_*(\mathbb{C}P^2)$:

$$0 \to \overset{4}{\mathbb{Z}} \to 0 \to \overset{2}{\mathbb{Z}} \to 0 \to \overset{0}{\mathbb{Z}} \to 0$$

$$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_{12} = \bigoplus_{i+j=12} C_i(X) \otimes C_j(Y) = \underset{8,4}{\mathbb{Z}} \text{ (Check other indices yield 0.)}$$

$$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_{10} = \bigoplus_{i+j=10} C_i(X) \otimes C_j(Y) = \underset{8,2}{\mathbb{Z}} \oplus \underset{6,4}{\mathbb{Z}} \text{ (Check other indices yield 0.)}$$

$$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_{8} = \bigoplus_{i+j=8} C_i(X) \otimes C_j(Y) = \underset{8,0}{\mathbb{Z}} \oplus \underset{6,2}{\mathbb{Z}} \oplus \underset{4,4}{\mathbb{Z}} \text{ (Check other indices yield 0.)}$$

$$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_{6} = \bigoplus_{i+j=6} C_i(X) \otimes C_j(Y) = \underset{6,0}{\mathbb{Z}} \oplus \underset{4,2}{\mathbb{Z}} \oplus \underset{2,4}{\mathbb{Z}} \text{ (Check other indices yield 0.)}$$

$$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_{4} = \bigoplus_{i+j=4} C_i(X) \otimes C_j(Y) = \underset{4,0}{\mathbb{Z}} \oplus \underset{2,2}{\mathbb{Z}} \oplus \underset{0,4}{\mathbb{Z}} \text{ (Check other indices yield 0.)}$$

$$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_{2} = \bigoplus_{i+j=2} C_i(X) \otimes C_j(Y) = \underset{2,0}{\mathbb{Z}} \oplus \underset{0,2}{\mathbb{Z}} \text{ (Check other indices yield 0.)}$$

$$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_{0} = \bigoplus_{i+j=0} C_i(X) \otimes C_j(Y) = \underbrace{\mathbb{Z} \otimes \mathbb{Z}}_{0,0} \cong \mathbb{Z} \text{ (Technically all } \mathbb{Z}\text{'s above are } \mathbb{Z} \otimes \mathbb{Z} \cong \mathbb{Z})$$

$(\mathbb{C}P^4 \otimes \mathbb{C}P^2)_n = 0$ for all other n.

The final computation, that all other chain groups are zero tells us that $\partial_i^{\otimes} = 0$ for all i i.e. all the differentials are zero. This tells us that we have already computed the Homology groups. For example, $\dfrac{\ker(\partial_8^{\otimes})}{\mathrm{Im}(\partial_9^{\otimes})} \cong \dfrac{\mathbb{Z}^3}{0} = \mathbb{Z}^3$. Therefore we get

$$H_k(\mathbb{C}P^4 \times \mathbb{C}P^2) = \begin{cases} \mathbb{Z} & \text{if } k = 0, 12 \\ \mathbb{Z}^2 & \text{if } k = 2, 10 \\ \mathbb{Z}^3 & \text{if } k = 4, 6, 8 \\ 0 \text{ else} & \end{cases}$$

16. Introduction to Cohomology Ring

Friday April 24th:

"What really matters is the work. And what matters to me is doing the work. I'm not looking at the back end: "What am I going to get out of this? What's going to be the reward?" I'm just looking at the work, the pleasure of being able to do the work."

-Robert Redford.

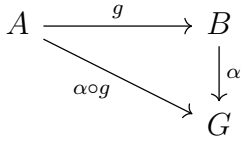
Before we can define this object, we need to understand cohomology *groups* first. Then by taking the direct sum of the cohomology groups and defining the cup product we arrive at the cohomology ring. Even though I will define everything for $\mathbb{Z}$-modules, you can verbatim replace it with an R-module (for a commutative ring with unity) and everything goes through.

Definition 39. Let A be a $\mathbb{Z}$ module and G any abelian group. Then the set

$$\mathrm{Hom}(A;G) := \{f : A \to G \mid f(a+a') = f(a) + f(a') \text{ for all } a, a' \in A\}$$

is in fact a $\mathbb{Z}$-module. The rule of addition is $(f+g) : A \longrightarrow G$ is defined as the function $(f+g)(a) = f(a) + g(a)$ and the rule of multiplication is given by $z \cdot f := A \longrightarrow G$ by $(z \cdot f)(a) = z \cdot f(a)$. If we let $G = \mathbb{Z}$ then the set $A* = \mathrm{Hom}(A;\mathbb{Z})$ is called the dual of A.

How can we combine maps from different Hom sets? Let A and B be $\mathbb{Z}$-modules and G an abelian group. Let $g : A \longrightarrow B$ be a homomorphism and $\alpha \in \mathrm{Hom}(B;G)$ and notice that we get following commutative diagram



Thus g induces a map, denoted by g^*, on the Hom sets, that is

$$g^* : \mathrm{Hom}(B;G) \longrightarrow \mathrm{Hom}(A;G)$$
$$g^*(\alpha) \longrightarrow \alpha \circ g$$

This allows us to go from a chain complex of $\mathbb{Z}$-modules to another chain complex of $\mathbb{Z}$-modules, except with arrows going the other direction (sometimes called a "co-chain"). Without defining it for a fixed abelian group G, $\mathrm{Hom}(-;G)$ defines a contravariant functor from complexes of $\mathbb{Z}$-modules to complexes of $\mathbb{Z}$-modules. Start with a chain complex of $\mathbb{Z}$ modules

$$\cdots \to C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$$

I need to tell you what $\mathrm{Hom}(-;G)$ does to the groups in the complex and what does it do to the maps ∂_*. The groups are clear, they go to $\mathrm{Hom}(C_*;G)$. For ease of notation, we define

$$\mathrm{Hom}(\partial_n;G) := \partial_n^* : \mathrm{Hom}(C_{n-1};G) \longrightarrow \mathrm{Hom}(C_n;G)$$
$$\partial_n^*(f : C_{n-1} \to G) \longrightarrow f \circ \partial_n : C_n \to G.$$

and now the complex becomes

$$\cdots \leftarrow \mathrm{Hom}(C_n;G) \xleftarrow{\partial_n^*} \mathrm{Hom}(C_{n-1};G) \xleftarrow{\partial_{n-1}^*} \mathrm{Hom}(C_{n-2};G) \leftarrow \cdots \leftarrow \mathrm{Hom}(C_1;G) \xleftarrow{\partial_1^*} \mathrm{Hom}(C_0;G) \leftarrow 0$$

To distinguish this from homology groups we denote the cochain complex (or the complex with arrows in the increasing direction!) as $C^* = (C_i^*, \partial_i^*)$ where $C_i^* = \mathrm{Hom}(C_i; G)$ and $\partial_i^* = \mathrm{Hom}(\partial_i; G)$. You can think of it as the superscript saying "the degree is going up i.e. from lower degree to higher".

Definition 40 (Cohomology of a chain)**.** Given a cochain complex we define its cohomology in degree n as

$$H^n(C^*; G) := \frac{\ker(\partial_{n+1})}{\mathrm{Im}(\partial_n)}$$

Example 51. Let us compute the Hom group $(\mathbb{Z}^m)^* := \mathrm{Hom}(\mathbb{Z}^m; \mathbb{Z})$ and show it is isomorphic to $\mathbb{Z}^m$. Let $e_1, \ldots, e_m$ be a basis of $\mathbb{Z}^m$ and define $e_k^* \in (\mathbb{Z}^m)^*$ as the delta-Kronecker function, that is $e_k^* : \mathbb{Z}^m \to \mathbb{Z}$ as $e_k^*(e_j) = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ else.} \end{cases}$ Now one can check that e_k^*'s generate $\mathrm{Hom}(\mathbb{Z}^m; \mathbb{Z})$ as a $\mathbb{Z}$-module and that the mapping $e_k \to e_k^*$ extends to an isomorphism between $\mathbb{Z}^m$ and $(\mathbb{Z}^m)^*$.

Example 52. Consider the following chain complex

$$0 \xrightarrow{0} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{0} 0$$

where its homology is given by

$$H_i = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z}_2 & \text{if } i = 1 \\ 0 & \text{if } i = 2 \\ \mathbb{Z} & \text{if } i = 3 \\ 0 & \text{else.} \end{cases}$$

Applying $\mathrm{Hom}(-, \mathbb{Z})$ and using the previous example, we get the groups are still $\mathbb{Z}$ and

$$0 \xleftarrow{0} Hom(\mathbb{Z}, \mathbb{Z}) \xleftarrow[Hom(0,\mathbb{Z})]{} Hom(\mathbb{Z}, \mathbb{Z}) \xleftarrow[Hom(\times 2,\mathbb{Z})]{} Hom(\mathbb{Z}, \mathbb{Z}) \xleftarrow[Hom(0,\mathbb{Z})]{} Hom(\mathbb{Z}, \mathbb{Z}) \xleftarrow{0} 0$$

Recall that $\mathrm{Hom}(h, \mathbb{Z})(f : \mathbb{Z} \to \mathbb{Z}) = f \circ h$. So all the maps $\mathrm{Hom}(0, \mathbb{Z})$ are indeed the zero map as composition with the 0 map yields the 0 map. Then $\mathrm{Hom}(2, \mathbb{Z})(f : \mathbb{Z} \to \mathbb{Z}) = f \circ 2 = f(2 \cdot z) = 2 \cdot f(z) = 2f$ thus the map $\mathrm{Hom}(2, \mathbb{Z})$ is still just multiplication by 2.

$$0 \xleftarrow{0} \underbrace{\mathbb{Z}}_{C^3} \xleftarrow{0} \underbrace{\mathbb{Z}}_{C^2} \xleftarrow{\times 2} \underbrace{\mathbb{Z}}_{C^1} \xleftarrow{0} \underbrace{\mathbb{Z}}_{C^0} \xleftarrow{0} 0$$

and now the cohomology groups are

$$H^i = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i = 1 \\ \mathbb{Z}_2 & \text{if } i = 2 \\ \mathbb{Z} & \text{if } i = 3 \\ 0 & \text{else.} \end{cases}$$

The content of the next lemma basically says that applying $\mathrm{Hom}(-,\mathbb{Z})$ translates to taking transpose.

Lemma 19. *If $A:\mathbb{Z}^m\to\mathbb{Z}^n$ has the matrix A with respect to basis $e_1,\dots,e_m$ on $\mathbb{Z}^m$ and the basis $f_1,\dots,f_n$ on $\mathbb{Z}^n$, then $\mathrm{Hom}(A,\mathbb{Z}):\mathrm{Hom}(A,\mathbb{Z}^n)=(\mathbb{Z}^n)^*\to\mathrm{Hom}(A,\mathbb{Z}^m)=(\mathbb{Z}^m)^*$ has the matrix $(A^*)_{m\times n}=A^T$ with respect to the dual basis $e_1^*,\dots,e_m^*$ and $f_1^*,\dots,f_n^*$.*

We can now take any of our earlier constructions of homology and replace with cohomology with coefficients in G that is

- $C_*^{\mathrm{sing}}(X)$-singular chain complex $\longrightarrow H^*_{\mathrm{sing}}(X;G)$ Singular cohomology
- $C_*^{\Delta}(X)$- Δ-chain complex $\longrightarrow H^*_{\Delta}(X;G)$ Δ-cohomology
- $C_*^{CW}(X)$-Cell complexes $\longrightarrow H^*_{\mathrm{CW}}(X;G)$ Cellular cohomology

Example 53. Let K be the Klein bottle and recall its Δ-complex structure:
The Δ-complex is given by

$$0\longrightarrow\mathbb{Z}\langle U,L\rangle\xrightarrow{\delta_2}\mathbb{Z}\langle a,b,c\rangle\xrightarrow{\delta_1=0}\mathbb{Z}\langle v\rangle\xrightarrow{\delta_0}0$$

where $\delta_2(U)=b-c+a$ and $\delta_1(L)=a-b+c$. This, and the fact that $\delta_1=0$ implies $\delta_1=[0,0,0]$ and $\delta_2=\begin{bmatrix}1&1\\1&-1\\-1&1\end{bmatrix}\sim\begin{bmatrix}2&0\\0&1\\0&0\end{bmatrix}$ which yielded the following homology groups

$$H_i(X)=\begin{cases}\mathbb{Z} & \text{if } i=0\\ \mathbb{Z}\oplus\mathbb{Z}_2 & \text{if } i=1\\ 0 & \text{if } i=2\end{cases}$$

Applying $\mathrm{Hom}(-,\mathbb{Z})$ we now get

$$0\longleftarrow\mathbb{Z}\langle U^*,L^*\rangle\xleftarrow{\delta_2^*}\mathbb{Z}\langle a^*,b^*,c^*\rangle\xleftarrow{\delta_1^*=0}\mathbb{Z}\langle v^*\rangle\xleftarrow{\delta_0}0$$

where $\delta_1^*=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and $\delta_2^*=\begin{bmatrix}1&1&-1\\1&-1&1\end{bmatrix}\sim\begin{bmatrix}2&0&0\\0&1&0\end{bmatrix}$. Thus the cohomology groups are

$$H^*_{\Delta}(K;\mathbb{Z})=\begin{cases}\mathbb{Z} & \text{if } i=0\\ \mathbb{Z} & \text{if } i=1\\ \mathbb{Z}_2 & \text{if } i=2\\ 0 & \text{else}\end{cases}$$

Monday April 27th:

"I'm interested in that thing that happens where there's a breaking point for some people and not for others. You go through such hardship, thing that are almsot impossibly difficult, and there's no sign that it's going to get any better, and that's the point when people quit. But some don't. "

-Robert Redford.

I will again assume A and B are $\mathbb{Z}$-modules for simplicity but you can easily replace them with R-modules for any commutative ring with unity. (Even commutative isn't necessary at which point we need the appropiate left/right R-module.)

Definition 41 ($\mathrm{Ext}(A,B)$)**.** Let

$$0 \to F_1 \xrightarrow{h} F_0 \to A \to 0$$

be the free resolution of A, described in more detail in Remark 14. Applying $\mathrm{Hom}(-,B)$ to the resolution yields

$$\mathrm{Hom}(F_1,B) \xleftarrow{h^*} \mathrm{Hom}(F_0,B) \leftarrow \mathrm{Hom}(A,B) \leftarrow 0. \tag{16.1}$$

Notice that we can make the the sequence in Equation (16.1) exact by adding the cokernel of H^*, that is the following sequence is exact:

$$0 \leftarrow \mathrm{coker}(h^*) \leftarrow \mathrm{Hom}(F_1,B) \xleftarrow{h^*} \mathrm{Hom}(F_0,B) \leftarrow \mathrm{Hom}(A,B) \leftarrow 0.$$

The group $\mathrm{coker}h^*$ is defined to be $\mathrm{Ext}(A,B)$.

We can immidiately observe that $\mathrm{Ext}(A,B)$ is again a $\mathbb{Z}$-module as it is the quotient of a $\mathbb{Z}$-module, namely $\mathrm{Hom}(F_1,B)$ by another $\mathbb{Z}$-module, namely $\mathrm{Im}(h^*)$. In particular, $\mathrm{Ext}(A,B)$ has the structure of a group.

One way to think about this definition is that $\mathrm{Ext}(A,B)$ "measure the lack of exactness of the dualized complex." I never introduced this terminology but the complex we get after applying $\mathrm{Hom}(-,G)$ is called the dual of the initial complex and applying $\mathrm{Hom}(-,G)$ is sometimes referred to as "dualizing the complex".

Remark 16. Technically what we have defined here is called the "first Ext group" as it is computed in degree 1. Moreover, when we write a free resolution of A i.e.

$$0 \to F_1 \xrightarrow{h} F_0 \to A \to 0$$

the resolution part doesn't include A. So we are applying $\mathrm{Hom}(-;G)$ the everything that is not A. (Although you can write it if you wish and sometimes it computes to zero anyways like the next example.)

Example 54. Let us compute $\mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z})$. Consider the free resolution of $\mathbb{Z}_n$ given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}_n \to 0.$$

Applying $\mathrm{Hom}(-;\mathbb{Z})$ we get

$$\mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \xleftarrow{\mathrm{Hom}(\cdot n,\mathbb{Z})} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \leftarrow \underbrace{\mathrm{Hom}(\mathbb{Z}_n,\mathbb{Z})}_{0} \leftarrow 0$$

$$\mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z} \leftarrow 0 \leftarrow 0$$

Now note that $\mathrm{coker}(\mathbb{Z} \xleftarrow{\cdot n} \mathbb{Z}) = \mathbb{Z}_n$ therefore $\mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z}) \cong \mathbb{Z}_n$.

Example 55. Let us compute $\mathrm{Ext}(\mathbb{Z}_m,\mathbb{Z}_n)$. Note that

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}_m \to 0$$

is a free resolution of $\mathbb{Z}_m$, to which we apply $\mathrm{Hom}(-,\mathbb{Z}_n)$ and get

$$0 \leftarrow \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_n) \xleftarrow{\mathrm{Hom}(\cdot m,\mathbb{Z}_n)} \mathrm{Hom}(\mathbb{Z},\mathbb{Z}_n) \leftarrow \underbrace{\mathrm{Hom}(\mathbb{Z}_m,\mathbb{Z}_n) \leftarrow 0}_{\text{irrelevant part}}$$

One can show that $\mathrm{Hom}(\mathbb{Z},\mathbb{Z}_n) \cong \mathbb{Z}_n$ and $\mathrm{Hom}(\mathbb{Z}_m,\mathbb{Z}_n) \cong \mathbb{Z}_d$ where $d = \gcd(m,n)$. Moreover, similar to previous computations we did in class, $\mathrm{Hom}(\cdot m,\mathbb{Z}_n)$ is just multiplication by m thus the complex becomes (more precisely is isomorphic to)

$$0 \leftarrow \mathbb{Z}_n \xleftarrow{\cdot m} \mathbb{Z}_n \leftarrow \mathbb{Z}_d \leftarrow 0$$

What's left is to compute cokernel$(\cdot m)$. Let $n = k_1 d$, $m = k_2 d$ and $d = sn + tm$ for integers s and m. First the image of multiplication by m is generated by $[m]_n$. I first claim that $\langle [m]_n \rangle = \langle [d]_n \rangle$. The inclusion $\subseteq$ is clear as $m = k_2 d$. The other inclusion is also clear since

$$d = sn + tm \equiv tm(\mod n).$$

Thus $\mathrm{Im}(\cdot) = \langle [m]_n \rangle = \langle [d]_n \rangle$, however note that $|\langle [d]_n \rangle| = k_1$ and thus $\left| \dfrac{\mathbb{Z}_n}{\langle [d]_n \rangle} \right| = \frac{n}{k_2} = d$ and since $\dfrac{\mathbb{Z}_n}{\langle [d]_n \rangle}$ is cyclic we get that $\dfrac{\mathbb{Z}_n}{\langle [d]_n \rangle} \cong \mathbb{Z}_d$. Therefore $\mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z}_m) \cong \mathbb{Z}_d$.

Proposition 6. *Let H be a finitely generated abelian group and G any abelian group. Then*

(1) $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$
(2) $\mathrm{Ext}(H,G) = 0$ *if H is free*
(3) $\mathrm{Ext}(\mathbb{Z}_n, G) \cong G/n \cdot G$

Both of the previous examples can be directly deduced from item 3.

Theorem 16.1 (Universal Coefficient Theorem for Cohomology)**.** *If (C_*, ∂_*) is a chain complex of free abelian groups with homology $H_*(C_*)$ then the cohomology of the dual cochain complex $(\mathrm{Hom}(C_*;G); \partial^*)$ is determined by the split short exact sequence*

$$0 \to \mathrm{Ext}(H_{n-1}(C_*), G) \to H^n(C_*;G) \xrightarrow{h} \mathrm{Hom}(H_n(C_*), G) \to 0.$$

The map h in the statement of Theorem 16.1 is defined as follows: an element $\phi \in H^n(C;G)$ is an equivalence class of a map $\phi : C_n \to G$. Applying h simple sends ϕ to the equivalence class of its restriction to Z_n i.e. $h([\phi]) = [\phi_{Z_n}]$. The well-definedness of this map is justified in Hatcher page 191. Moreover, h is a surjective homomorphism.

Note that if $\mathrm{Ext}(H_{n-1}(C_*), G) \cong 0$ then h becomes an isomorphism and we get

$$H^n(C_*;G) \cong \mathrm{Hom}(H_n;G)$$

Corollary 11. *If $H_n(C_*)$ and $H_{n-1}(C_*)$ are finitely generated such that $H_n \cong \mathbb{Z}^{\beta_n} \oplus T_n$ and $H_{n-1} \cong \mathbb{Z}^{\beta_{n-1}} \oplus T_{n-1}$ then*

$$H^n(\mathbb{C}^*, \delta^*; \mathbb{Z}) \cong \mathbb{Z}^{\beta_n} \oplus T_{n-1}.$$

Example 56. Consider the Klein bottle K. Recall that

$$H_k(K) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } k = 1 \\ 0 & \text{else} \end{cases} \Rightarrow H^k(K) = \begin{cases} \mathbb{Z} & \text{if } k = 0 \\ \mathbb{Z} \oplus \underbrace{T_0}_{0} & \text{if k=1} \\ 0 \oplus \underbrace{T_1}_{\mathbb{Z}_2} & \text{if } k = 2 \\ 0 & \text{if } k > 2 \end{cases}$$

Recall that when we defined homology we defined something called the total homology group which was the direct sum of all homology groups i.e.

$$H_*(C_*, \partial_*) = \bigoplus_{k \geq 0} H_k(C_*, \partial_*).$$

Similarly, the total cohomology group is defiend as

$$H^*(C^*, \partial^*) = \bigoplus_{k \geq 0} H^k(C^*, \partial^*)$$

For example, the total cohomology for the Klein bottle is $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2$. Turns out that the cohomology group is endowed with a product which makes it a ring. This is called the cup product.

Definition 42. There is a bilinear map on singular cochains as follows:

$$\smile : C^k(X) \times C^l(X) \longrightarrow C^{k+l}(X)$$

$$(\underbrace{\phi}_{\in \mathrm{Hom}(C_k(X);\mathbb{Z})} , \underbrace{\psi}_{\in \mathrm{Hom}(C_k(X);\mathbb{Z})}) \longrightarrow \phi \smile \psi \in \mathrm{Hom}(C_{k+l}(X); \mathbb{Z})$$

$$\text{where } (\phi \smile \psi)(\tau : \Delta^{k+l} \to X) = \phi(\tau|_{[v_0,\ldots,v_k]}) \cdot \psi(\tau|_{[v_k,\ldots,v_{k+l}]})$$

Multiplication in $\mathbb{Z}$

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